Introduction

• Measurement planning

- Some everyday examples — Temperature outside
  - Weight of the body

indirect quant.
accuracy

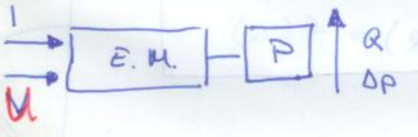
1 device → immediately obtained results

Why is it important measurement planning?

- In engineering the situation is more complicated!

Example 1

Measurement of the efficiency of an electric motor-pump ~~tartar~~ group



$\boxed{\text{Efficiency } (\eta) \ ?}$

there is no device to measure directly!

$$\eta = \frac{\overset{\text{useful}}{P_u}}{\underset{\text{input}}{P_i}} = \frac{\Delta p \cdot Q}{I \cdot U}$$

η — P_u — Δp, Q

η — P_i — U, I

Δp, Q, U, I } variables.

Determine the required variables and the possible parameters

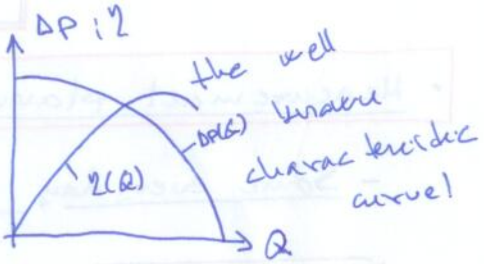
n - revolution number
T - liquid temperature, $\mu = f(T)$ viscosity
f_s - electric network frequency ~50 Hz in case of electric devices.
⋮
} Parameters

$$\boxed{\eta = f(\Delta p, Q, V, I; n, T, f_s \ldots)}$$

Dependent / Independent variables

$\Delta p = f(Q)$

$Q = f(\Delta p)$



U - constant, independent

$I = f(\Delta p, Q)$

How many ~~independen~~ ~~parameters~~ variables can describe the state of the system?

- Only 1, but 2 possible choice : Q or ΔP

Let us choose Q

↓

$\boxed{Q - \text{control variable}}$

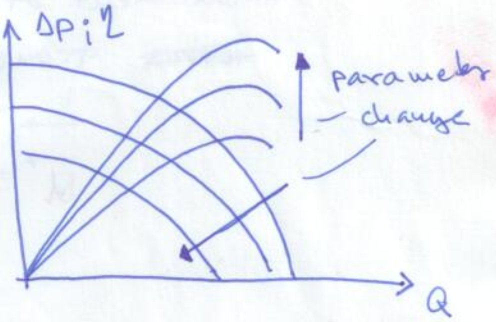
extraneous:

$\Delta P = f(Q)$

V - constant

$I = f(\Delta P(Q); Q) = f(Q)$

Determine the control variables



All the other variables are extraneous! (külső)

- We cannot or do not want to control.

Controlled / extraneous parameters

n - controlled

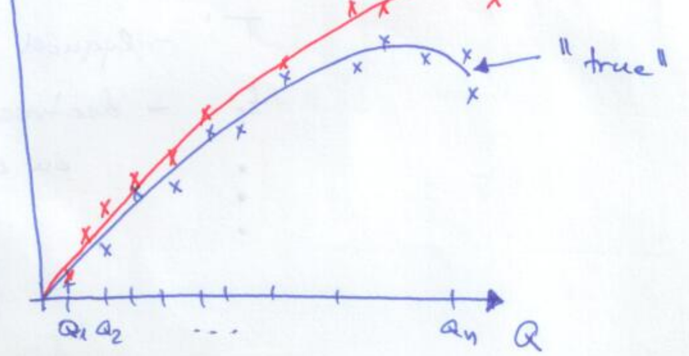
T, f_s } extraneous → Dangerous as they can cause false trends.

for instance

- Q changed continuously.
- T changed -"-

↓

Interference
(false trend)



Solution

- Make it controllable

  e.g. cooling device

- Random measurement

  Originally: Q was increased from Q_{min} to Q_{max} continuously! $(Q_1, Q_2, \dots, Q_n)$

  Random: $(Q_3, Q_{10}, Q_1, Q_5, Q_4, \dots Q_{n-6})$

  ↓

  break the trend

  ↓

  become random variation error like noise

  ↓

  $\boxed{\text{handled by statistics}}$

$\boxed{\text{Statistics}}$

- Every measured variable contain random variation (noise) or other kind of uncertainty.
- The true value of the desired variable X is not known.
- Estimation for X is:

$$\underbrace{\boxed{X \approx X' = \underbrace{\bar{X}}_{\text{average}} \pm \underbrace{u_x}_{\text{uncertainty}} \quad \underbrace{(P\%)}_{\text{confidence e.g. } 95\%}}}_{\text{estimation}}$$

- For dependent variables (for instance the efficiency)
  (complex)

$$y = f(x_1, x_2, \dots x_n)$$

$$\downarrow \quad u_{x_1} \quad u_{x_2} \quad \dots \quad u_{x_n}$$

$$\to u_y = ?$$

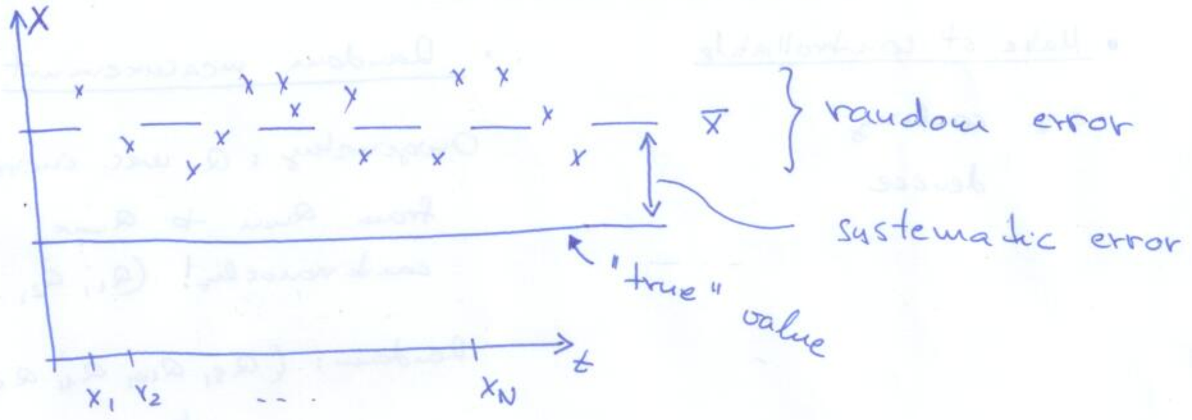
Error propagation!

Sensitivity analysis!

↓

Choice of measurement device!

- uncertainty of a single variable
(direct measurable)



- random error (confidence interval)

$$\boxed{u_r = \lambda_{st}\cdot\frac{s^*}{\sqrt{N}}}$$

λ_{st} – Student coefficient

$$\lambda_{st} = f(P;\ \underbrace{N-1}_{\text{degree of freedom}})$$

→ confidence / probability

s^* – standard deviation.

- sytematic error
  - calibration → $\boxed{u_s = \phi}$
  - if calibration is not possible:

data sheet: Pressure transducer,

Range: 1 – 10 bar (p_{max} = 10 bar)

Class of accuracy: $\varepsilon_{cal} = 0{,}5\,\%$ – usually in terms of p_{max}

↓

$$\boxed{u_s = \underset{(X_{max})}{p_{max}}\cdot\varepsilon_{cal}} - 10\ \text{bar}\cdot 0{,}005 = \underline{\underline{0{,}05\ \text{bar}}}$$

$$\boxed{u_x = \sqrt{u_r^2 + u_s^2}}$$

- other errors similarly:

Linearity: ε_{lin}

Hysteresis: ε_{hys}

⋮

$$\boxed{u_x = \sqrt{u_r^2 + u_s^2 + u_{lin}^2 + u_{hys}^2 + \dots}}$$

- uncertainty of indirect variables

$$y = f(x_1, x_2, \dots, x_n)$$

Error propagation:

$$\boxed{u_y^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \cdot u_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \cdot u_{x_2}^2 + \dots + \left(\frac{\partial f}{\partial x_n}\right)^2 \cdot u_{x_n}^2}$$

$\frac{\partial f}{\partial x_i}$ - sensitivity coefficient!

if $\partial f / \partial x_i$ big → small error will be magnified!

↓

more precise instrument is needed!

Example 2

Density measurement of a cylinder

$$\rho = \frac{m}{V}, \quad V = \frac{d^2 \pi}{4} \cdot H$$

$$\boxed{\rho = \frac{4 \cdot m}{d^2 \pi H}}$$

Variables, Data Sheet

$\varepsilon_m = 0.5\%$; $m_{max} = 5kg$

$\varepsilon_d = 0.5\%$; $d_{max} = 500 mm$

$\varepsilon_H = 0.5\%$; $H_{max} = 500 mm$

- measurement results

N=10 for each variable

$\overline{m} = 2990g$; $S_m^* = 18g$

$\overline{d} = 101.5 mm$; $S_d^* = 0.51 mm$

$\overline{H} = 402 mm$; $S_H^* = 1,3 mm$

P = 95%

$\lambda_{st} = f(N-1; P) = \underline{\underline{2,23}}$

$$\left.\begin{aligned} u_r^m &= 12.7g \\ u_s^m &= 25g \end{aligned}\right\} \underline{\underline{u_m = 28g}}$$

$$\underbrace{\frac{\partial \rho}{\partial m} = \frac{4}{d^2 \pi H}; \quad \frac{\partial \rho}{\partial d} = -\frac{8m}{d^3 \pi H}; \quad \frac{\partial \rho}{\partial H} = -\frac{4m}{d^2 \pi H^2}}$$

$$\left.\begin{aligned} u_r^d &= 0.36mm \\ u_s^d &= 2.5mm \end{aligned}\right\} u_d = \underline{\underline{2,52mm}}$$

$$\left(\frac{u_\rho}{\rho}\right)^2 = \left(\frac{u_m}{m}\right)^2 + \left(\frac{2 \cdot u_d}{d}\right)^2 + \left(\frac{u_H}{H}\right)^2 =$$

$$= 8,77e-5 + \boxed{246.6e-5} + 4.38e-5 =$$

$$\left.\begin{aligned} u_r^H &= 0.92mm \\ u_s^H &= 2.5mm \end{aligned}\right\} u_H = \underline{\underline{2.66mm}}$$

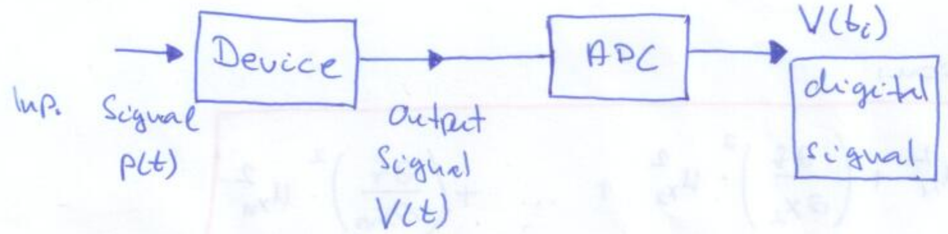
$$= 259.7e-5$$

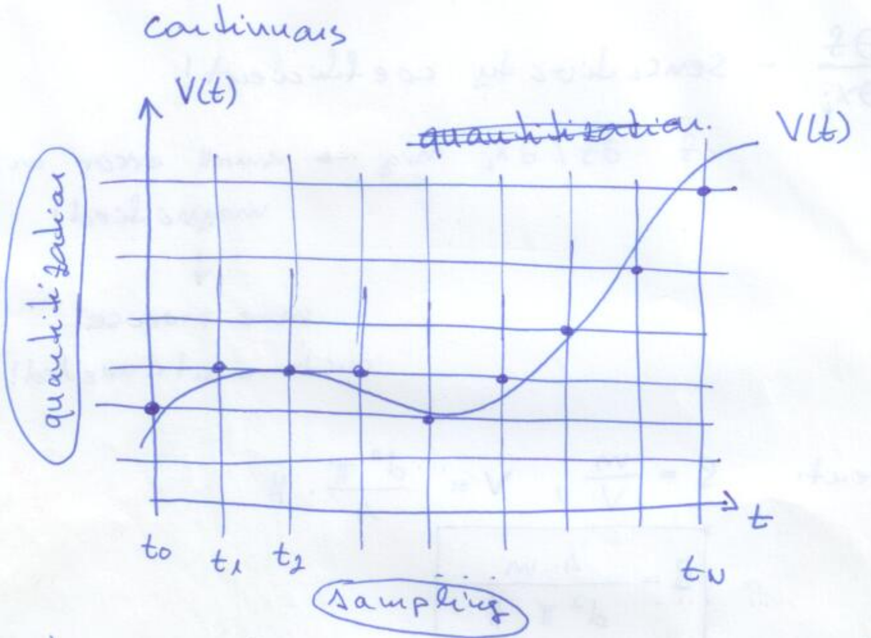
$$\frac{u_\rho}{\rho} = 0,05096 \rightarrow \boxed{\frac{u_\rho}{\rho} \cong 5,1\%}$$

Data Acquisition

②

- Problem





N - number of measurement

Questions: - How to choose the sample rate?

- What will be the resolution of signal V(t)? during quantitization.

↓

How can we extract the important feature of V(t)

- Quantitization error
  - Digital devices -> binary code

Unit: bit 0 or 1

Series of bit: word e.g. 10011010001 11-bit word.

specially: 8 bit word = byte

↓

8 bit sytem has 2^8 ~~two~~ = 256 different words

16 — " — 2^{16} = 65.536 — " —

32 — " — 2^{32} = 4.294.967.296 — " —

~~St~~ Generally

An M-bit system has 2^M different words!

↓

The full scale range $U_{FSR} = U_{max} - U_{min}$ is divided into ~~$Q = U_{FSR}$~~ $2^M - 1$ interval.

So the resolution: $$\boxed{Q = \frac{U_{FSR}}{2^M - 1} = LSB}$$

least significant bit

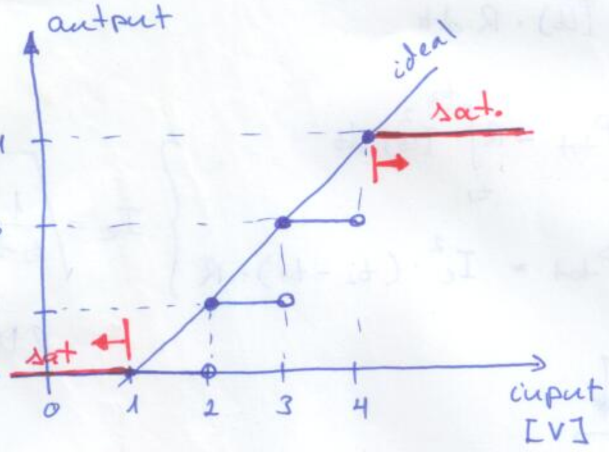
this is the least V change, which cause 1bit change in the output! 5p

Example 3

Presure tr. 1 - 4 V antput

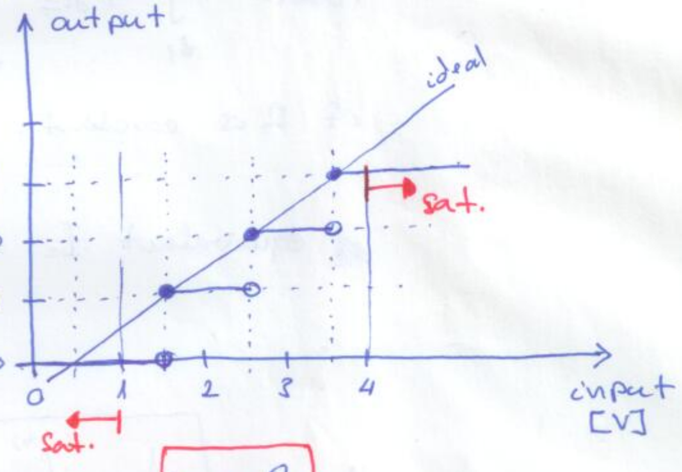
2bit ADC

$$Q = \frac{4-1}{2^2-1} = \frac{3}{3} = \underline{1}\ V \ ; \ 2^2 = 4 \text{ words}$$



$\boxed{U_q < Q}$

Shift with half Q



$\boxed{U_q \pm \frac{Q}{2}}$

10p

Example 4

U range: 0 - 10V

ADC: 8 bit

$U_i = 100mV$

$$Q = \frac{U_{FSR}}{2^8 - 1} = \frac{10V}{255} = \underline{\underline{39,2\ mV}}$$

$$U_q = \pm \frac{Q}{2} = \underline{\underline{19,6\ mV}}$$

5p

Relative Error: $\frac{U_q}{U_i} = \frac{19,6mV}{100mV} = \underline{19.6\%}$ BIG!!!

Solution: - ~~The~~ Increase the number of bits!

- Decrease U_{FSR}!? → smaller measurement range!

p ≅ mbar; do not use: p range 0 - 10bar!!

System Behaviour

- The measurement device is a dinamical system.

  ↳ Example: meaurement of body temperature!
  There is a response time.

- So far: Static calibration (infinitely fast response)

  Now: Dynamic calibration (Response characteristics for ditrent input signals)

- System model

  - Usually linearity is importat during measurement

    ↳ usually the system can be modelled with liner model.

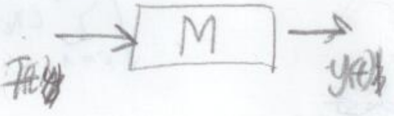
  - In general:

$$\boxed{a_n \frac{dy^n}{dt} + a_{n-1} \frac{dy^{n-1}}{dt} + \dots + a_1 \frac{dy}{dt} + a_0 y = F(t)}$$

↑ Input / "true" signal

a_i – system coeffizients.

- 3 cases



  - Zero-order System (infinit response time)
  - First-order (capacity)
  - Secont-Order (moment of inartia)

  } - Step function.
  - Sine —"—

• Zero - order system

$a_0 y = F(t)$

$y = K F(t)$; $\boxed{K = \frac{1}{a_0}}$ - static sensitivity / slope of the static calibration curve!

$\boxed{y = \cancel{K \cdot F}}$

- Infinite response time.
- For example: Static calibration.

e.g. pressure transd.

y [V]

F [bar]

K [V/bar]

• First - order system

$$\boxed{a_1 \cdot \frac{dy}{dt} + a_0 y = F(t)}$$

containing storage elements

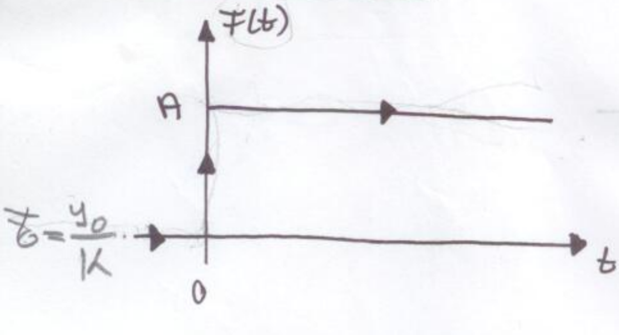
/: a_0 ; $\dot{\square} = \frac{d\square}{dt}$

$$\boxed{\tau \cdot \cancel{\frac{dy}{dt}}\dot{y} + y = K\, F(t)}$$

τ - time constant

K - static sensitivity

• <u>Step function:</u>



$$\boxed{\begin{array}{c}\tau \cdot \dot{y} + y = K \cdot A \\ y(0) = y_0\end{array}}$$

$t \geq 0$

Solution is the exponential function.

$$y_H = C \cdot e^{\lambda t} \quad ; \quad \dot{y}_H = \lambda \cdot C \cdot e^{\lambda t}$$

Hom: $\tau \lambda \cancel{C \cdot e^{\lambda t}} + \cancel{C e^{\lambda t}} = \emptyset$ - characteristic eq.

$\tau \lambda + 1 = \emptyset$

$$\boxed{\lambda = -\frac{1}{\tau}} \to \boxed{y_H = C \cdot e^{-\frac{1}{\tau} \cdot t}} \quad (\tau \dot{y} + y = 0)$$

Inhom: (Similar form than the excitation)

Polinomial -> Polinomical

trigonometric -> trigonometric.

$\dot{x} = x$

$$\boxed{y_{1H} = C_{1H}\cdot} \qquad C_{1H} = ?$$

$$\dot{y}_{1H} = \phi$$

Substitution into the equation:

$$\boxed{C_{1H} = KA}$$

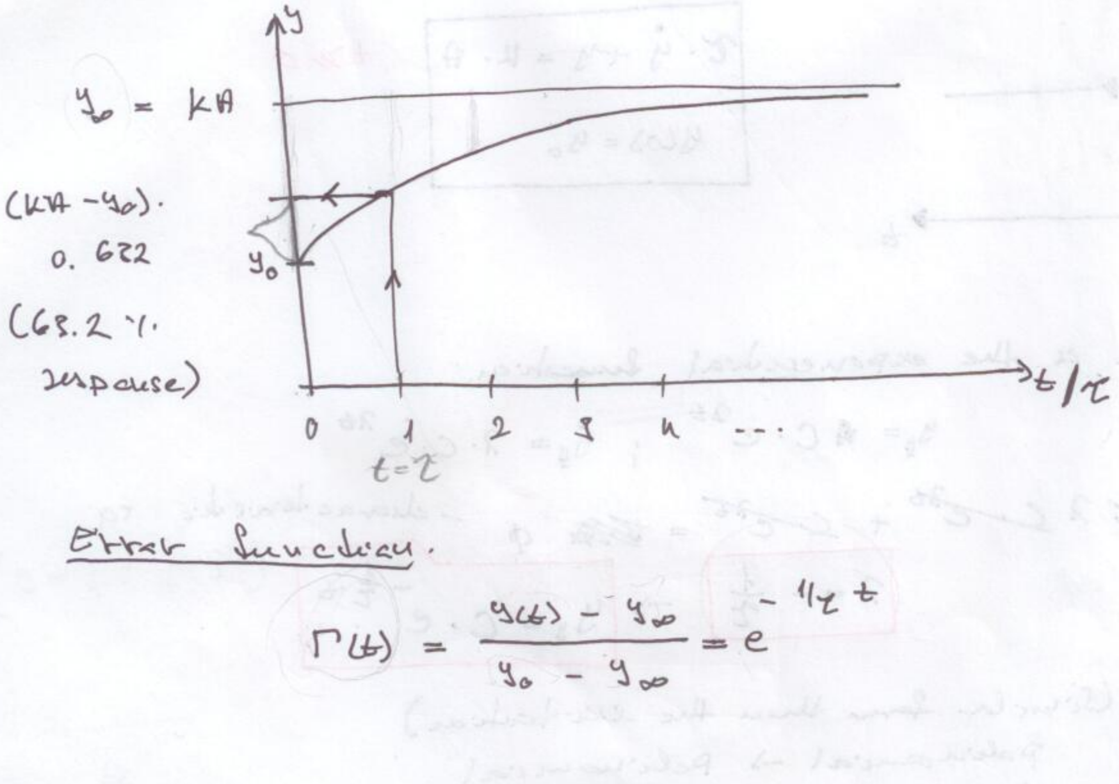
$$y = y_H + y_{1H}$$

$$y = KA + C\cdot e^{-\frac{1}{\tau}t}$$ – general solution

I.C. $y(0) = y_0$

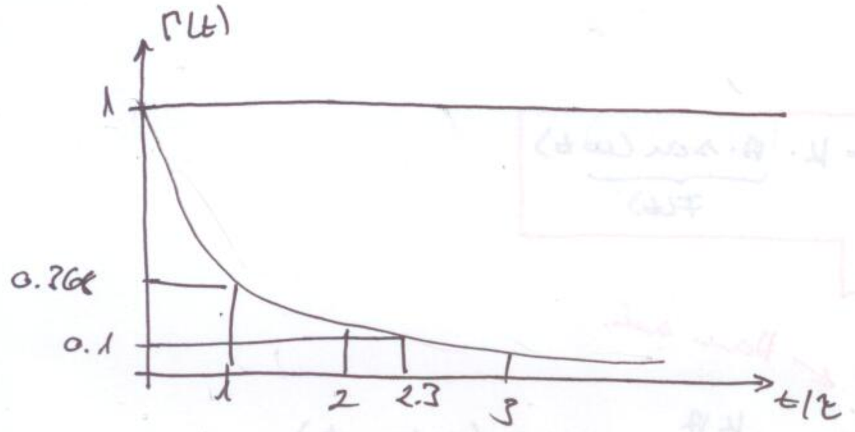
$$y_0 = K\cdot A + C\cdot \underbrace{e^{-\frac{1}{\tau}\cdot 0}}_{1} \rightarrow C = y_0 - KA$$

$$\boxed{y(t) = \underbrace{K\cdot A}_{\text{steady response}} + \underbrace{(y_0 - KA)\cdot e^{-1/\tau\cdot t}}_{\text{transcient response}}}$$



Error function.

$$\Gamma(t) = \frac{y(t) - y_\infty}{y_0 - y_\infty} = e^{-1/\tau\, t}$$



$t/\tau = 1 \rightarrow \Gamma(t) = 0.368 = 36.8\%$ → τ is how fast the system responses.

$= 2$ 0.135 13.5%

$= 3$ 0.05 5%

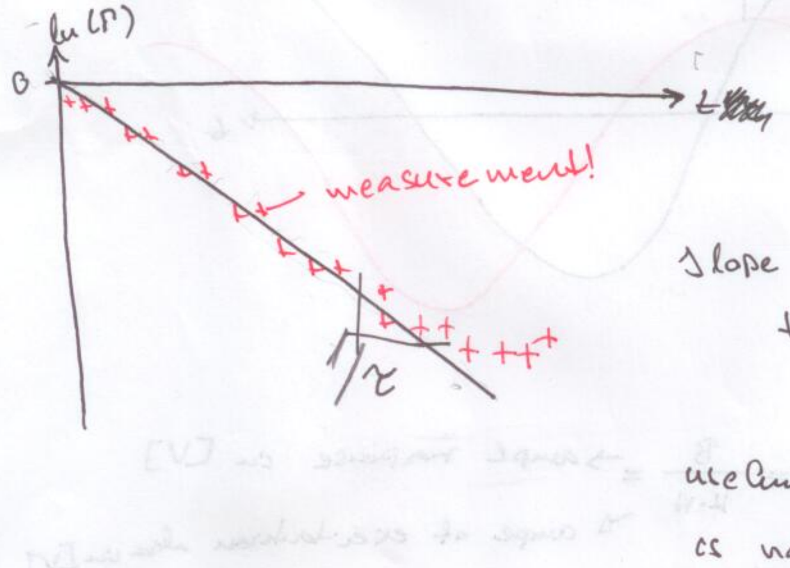
τ is the time needed to achieve 63.2% response.

≈ 2.3 0.1 90%.

↳ rise time of the system → $\boxed{t_r = 2.3 \cdot \tau}$

$$\Gamma(t) = e^{-1/\tau t} \quad / \ln()$$

$$\ln(\Gamma) = -\frac{1}{\tau} \cdot t$$ → linear in a semi log plot



Slope of the curve is the time constant

↓

useful, if the system is not linear → the measured points will deviate from the curve!

• Sine function input

$$\boxed{\tau\cdot\dot{y} + y = K\cdot\underbrace{A\cdot\sin(\omega t)}_{F(t)}}$$

I.C: $\boxed{y(0) = y_0}$

General ~~Specific~~ solution:

Hom. sol.

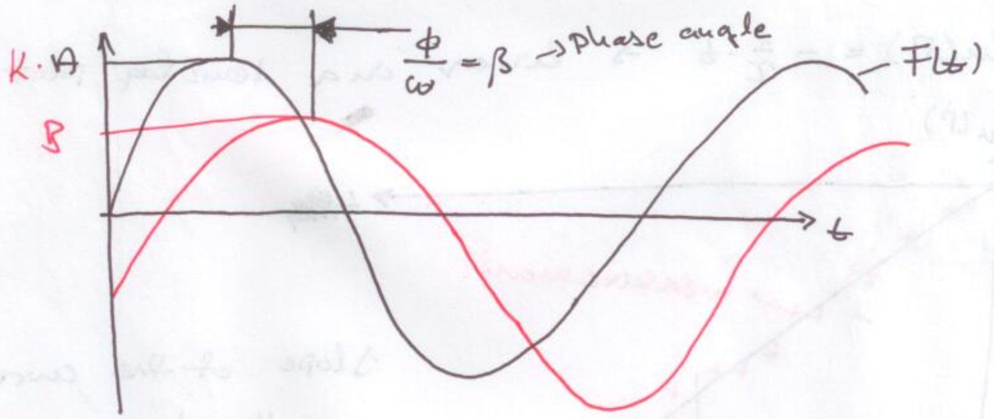
$$y(t) = \underset{\substack{\downarrow \\ \text{transient}}}{C\cdot e^{-\frac{t}{\tau}}} + \underbrace{\frac{KA}{\sqrt{1+(\omega\tau)^2}}\cdot\sin(\omega t - \phi)}_{\text{steady state}}$$

Long-term behaviour:

$$y_\infty(t) = B(\omega)\sin(\omega t - \phi(\omega)) = B(\omega)\sin\left(\left(t - \frac{\phi}{\omega}\right)\omega\right)$$

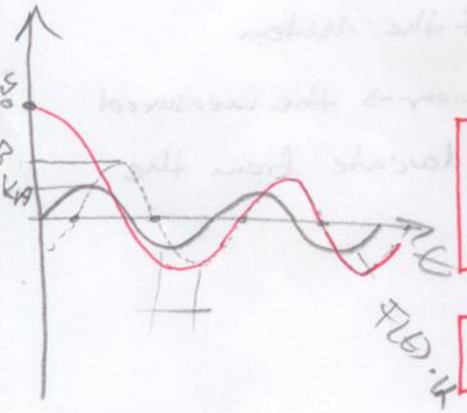
$B(\omega) = \dfrac{KA}{\sqrt{1+(\omega\tau)^2}}$ - amplitude of the stedy response

$\phi(\omega) = \tan^{-1}(\omega\tau)$ - phase shift



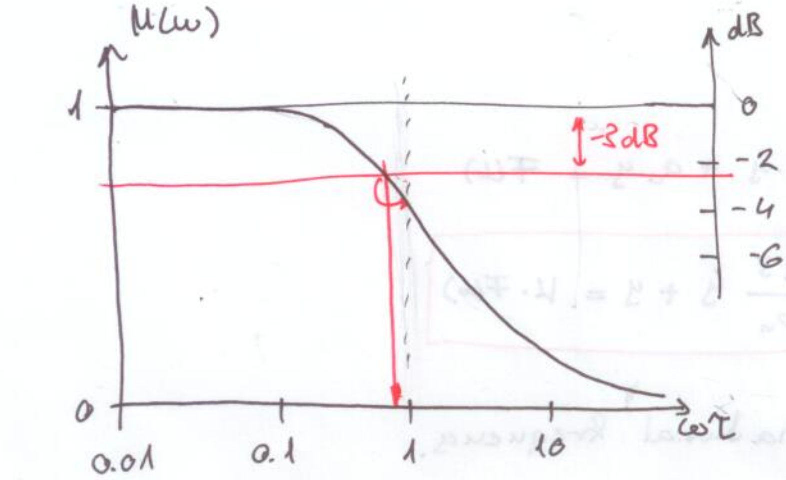
• Amplification:

$M(\omega) = \dfrac{B}{K\cdot A}$ → ampl response in [V]; → ampl. of excitation also in [V]



$$\boxed{M(\omega) = \frac{1}{\sqrt{1+(\omega\tau)^2}}}$$

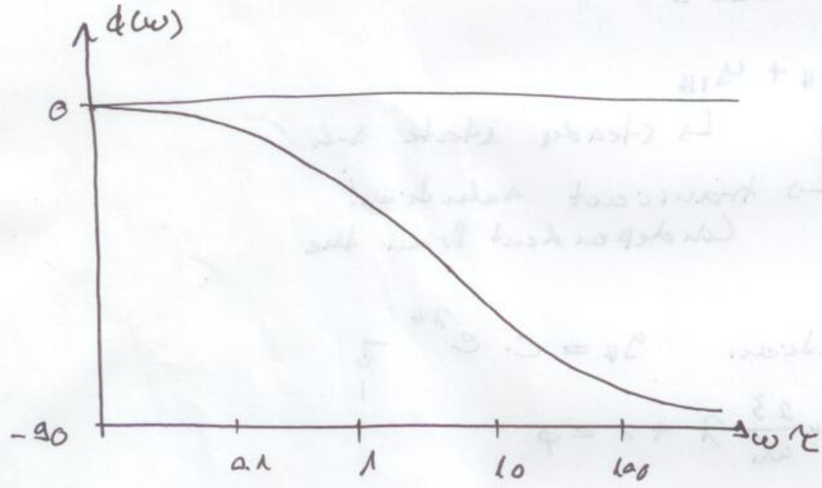
$$\boxed{\phi(\omega) = \tan^{-1}(\omega\tau)}$$



M(ω) << such frequency components are filtered!

M(ω) in dB = $\underline{\underline{20 \log M(\omega)}}$

Frequency Bandwidth $\boxed{\begin{array}{l} M(\omega) \geq -3\,dB \\ M(\omega) \geq 0.707 \end{array}} \rightarrow \boxed{\omega_b \tau = 1}$



- Phase linearity is importat, usually a signal contains multiple frequencies!

$\downarrow$

- Discussion in Second-Order systems!

• Second-Order system

$$a_2 \cdot \ddot{y} + a_1 \cdot \dot{y} + a_0 \cdot y = F(t)$$

$$\boxed{\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = K \cdot F(t)}$$

$\omega_n = \sqrt{\frac{a_0}{a_2}}$ – natural frequency.

$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}$ – damping ratio.

$K = \frac{1}{a_0}$ – static sensitivity.

I.C. $y(0) = y_0$; $\dot{y}(0) = \dot{y}_0$

Solution: $y(t) = y_H + y_{IH}$

↳ steady state sol.

↳ transcient solution.
(independent from the

• <u>Homogeneous sol.</u>

characteristic equation $y_H = C \cdot e^{\lambda t}$

$$\frac{1}{\omega_n^2}\lambda^2 + \frac{2\zeta}{\omega_n}\lambda + 1 = \phi$$

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \quad \sqrt{1-\zeta^2}\ ?$$

a) $0 \leq \zeta \leq 1$ (under damped system)

$$y_H(t) = C \cdot e^{-\zeta\omega_n t} \sin\Big(\overset{\omega_d}{\omega_n\sqrt{1-\zeta^2}}\, t + \phi\Big)$$

oscillatory behaviour

b) $\zeta > 1$ (overdamped system)

$$y_H(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

smooth response

c) $\zeta = 1$ (critically damped system)

$y_H(t) = C_1 e^{\lambda t} + C_2 t\, e^{\lambda t}$ divide a) and b)

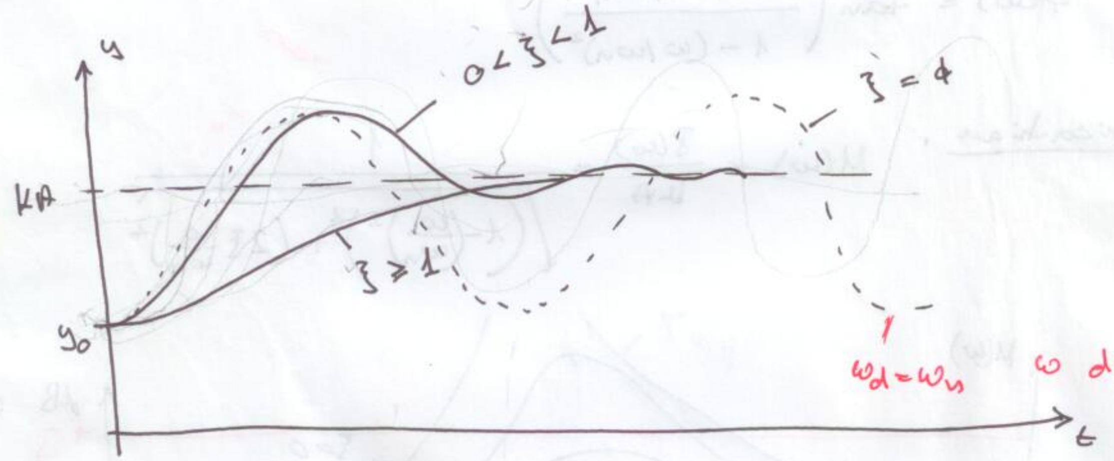
• Step function

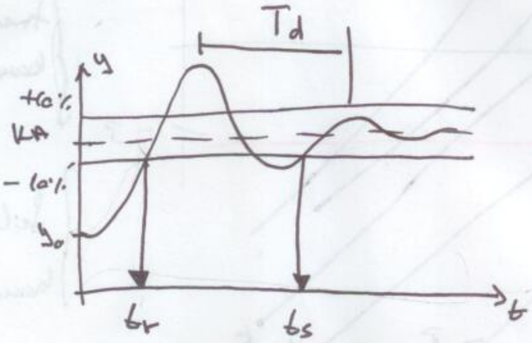
$F(t) = A \rightarrow \boxed{y_v = K \cdot A}$

$$y(t) = KA - KA \cdot e^{-\xi \omega_n t}\left[\frac{\xi}{\sqrt{1-\xi^2}} \sin\left(\omega_n \sqrt{1-\xi^2}\, t\right) + \cos\left(\omega_n \sqrt{1-\xi^2}\, t\right)\right]; 0 \leq \xi < 1$$

$$y(t) = KA - KA\left(1+\omega_n t\right) e^{-\omega_n t}; \quad \xi = 1$$

$$\underset{\downarrow \text{ Steady state}}{y(t) = KA} - \underbrace{KA\left[\frac{\xi+\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} - \frac{\xi-\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}\right]}_{\text{transcient response.}}; \xi > 1$$





$$T_d = \frac{2\pi}{\omega_d} = \frac{1}{f_d}$$

$$\boxed{\omega_d = \omega_n \sqrt{1-\xi^2}}$$

↳ ringing frequency / damped natural frequency!

Good compromise.

$\boxed{\xi = 0.7}$

In practice:

$0.6 \leq \xi \leq 0.8$

Rise time: $\boxed{t_r}$ • first reach 10% accuracy

Settling time: $\boxed{t_s}$ • stably reach 10% accuracy

• Sine function input

$$F(t) = A \cdot \sin(\omega t)$$

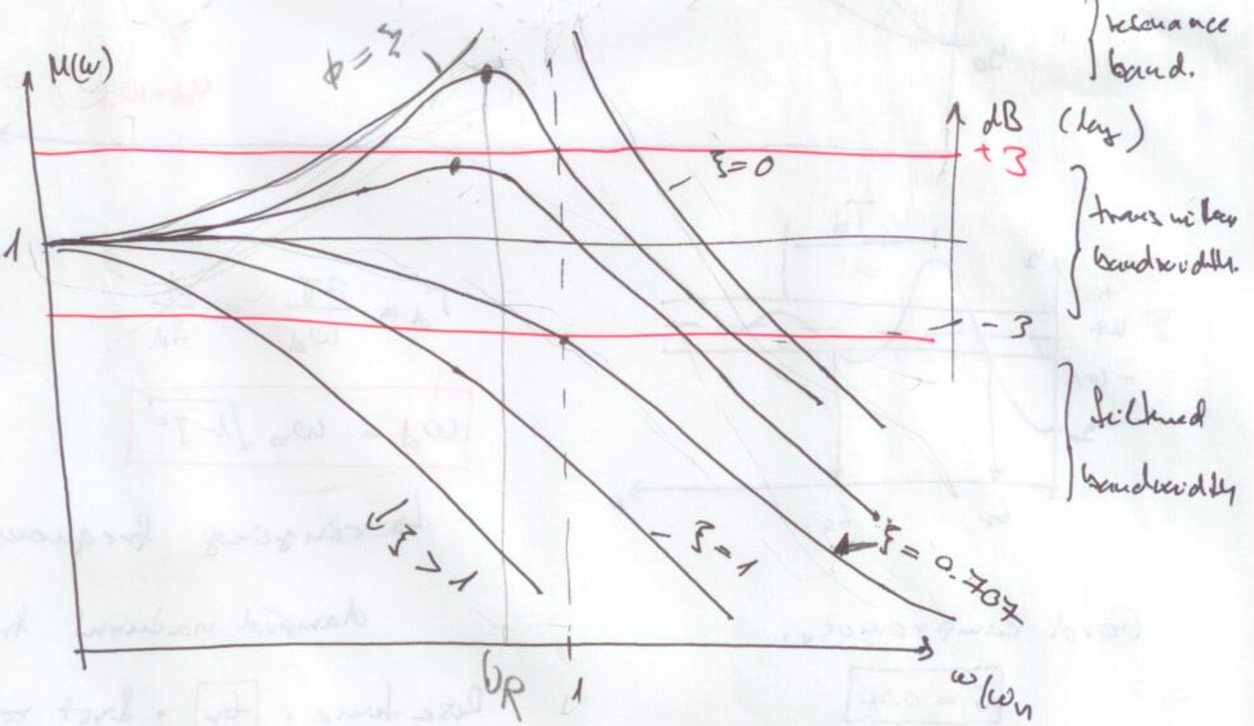
$$y(t) = y_h + \boxed{\frac{KA \cdot \sin(\omega t + \phi(\omega))}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}} = B(\omega)\sin(\omega t + \phi(\omega))$$

Stedy state response!

$$B(\omega) = \frac{KA}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\phi(\omega) = \tan^{-1}\left(\frac{2\xi\,\omega/\omega_n}{1-(\omega/\omega_n)^2}\right)$$

Amplification. $M(\omega) = \frac{B(\omega)}{KA} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$



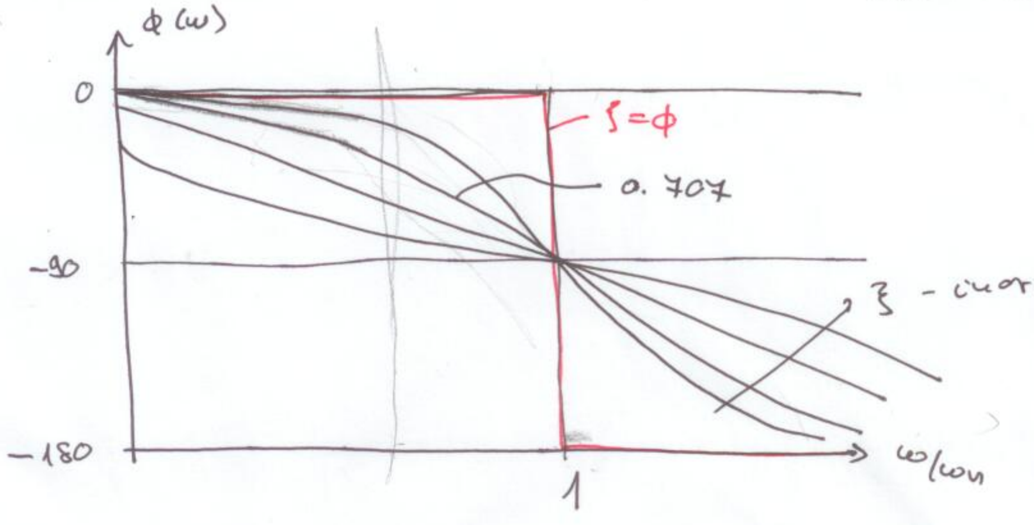
Peak resonant frequency: $\boxed{\omega_R = \omega_n\sqrt{1-2\xi^2}}$

if: $\xi = 0 \rightarrow \boxed{\omega_R = \omega_n}$ $2\xi^2 = 1$

$\boxed{\xi = \sqrt{0.5}} = 0.7071$

phase linearity

$u(t) = \sin(\omega_1 t) + \sin(\omega_2 t + \phi)$



- there is always phase shift!

- It can be a problem:

Example 9

- true signal $\quad u(t) = \sum_{n=1}^{\infty} \sin(n\omega t) =$ $\sin(\omega t) + \sin(2\omega t) + \dots$ $\sin(\omega_1 t) + \sin(2\omega_2 t) + \dots$

$\omega_n = n \cdot \omega$

- during the measurement there is a phase shift:

$$y(t) = \sin(\omega_1 t + \phi(\omega_1)) + \sin(2\omega_1 t + \phi(2\omega_1)) + \dots$$

- If $\phi(\omega)$ is linear:

$$y(t) = \sin(\omega_1 t + \phi^*) + \sin(2\omega_1 t + 2\phi^*) + \dots =$$

$$= \sin(\omega t + \phi) + \sin(2\omega t + 2\phi) + \dots$$

$$\theta = (\omega t + \phi)$$

$y(t) = \sin\theta + \sin 2\theta + \dots$ → the signal waveform is the same!!

the is no phase shift between the "true" and the measured signal!

$\boxed{\text{Fourier sor (series)}}$

Periodic function: $f(t)$; $f(t) = f(t+T)$

Fourier series: $f(t) = a_0 + \sum_{i=1}^{\infty} a_i \cos\left(i \frac{2\pi}{T} \cdot t\right) + b_i \sin\left(i \frac{2\pi}{T} \cdot t\right)$

$$a_0 = \frac{1}{T} \int_0^T f(t)\,dt \qquad \omega_i = i \cdot \frac{2\pi}{T}$$

$$a_i = \frac{2}{T} \int_0^T f(t) \cdot \cos\left(i \frac{2\pi}{T} t\right) dt \qquad \Delta\omega = \frac{2\pi}{T}$$

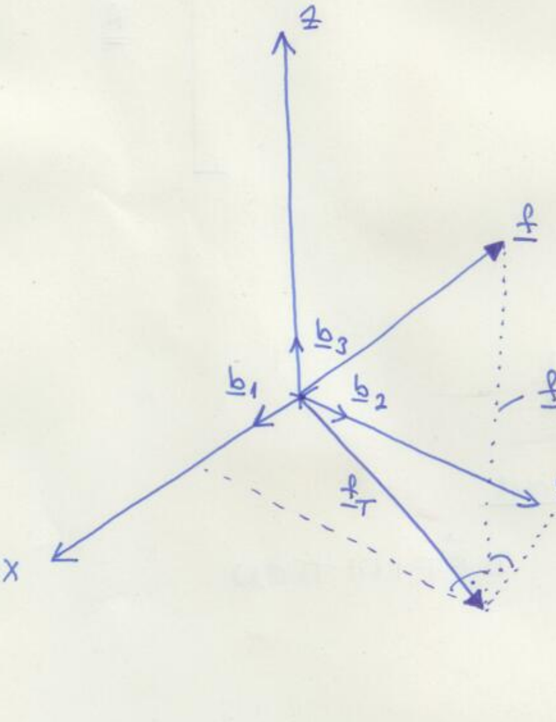
$$b_i = \frac{2}{T} \int_0^T f(t) \cdot \sin\left(i \frac{2\pi}{T} t\right) dt$$

In practice: $f_T(t) = a_0 + \sum_{i=1}^{N} a_i \cos\left(i \frac{2\pi}{T} t\right) + b_i \sin\left(i \frac{2\pi}{T} t\right)$

(truncated Fourier series)

How should a_0, a_i and b_i be determined?

Orthogonal projection.
(vector analogy)



Basis vectors: $\underline{b}_1, \underline{b}_2, \underline{b}_3$

Arbitrary vector: (in 3D)

$$\boxed{\underline{f} = (a_1 \underline{b}_1 + a_2 \underline{b}_2 + a_3 \underline{b}_3)}$$

Truncated vector: (in 2D):

$$\boxed{\underline{f}_T = a_1 \underline{b}_1 + a_2 \underline{b}_2} \quad \begin{matrix} a_1 = ? \\ a_2 = ? \end{matrix}$$

$\underline{f} - \underline{f}_T$ Best approximation

$\underline{f} - \underline{f}_T \perp \underline{b}_1$ and

$\underline{f} - \underline{f}_T \perp \underline{b}_2$

$$\left.\begin{aligned} \langle \underline{f} - \underline{f}_T, \underline{b}_1 \rangle = 0 \\ \langle \underline{f} - \underline{f}_T, \underline{b}_2 \rangle = 0 \end{aligned}\right\} \text{eq. sys.}$$

scalar product: $\langle .,. \rangle$ $\quad \langle \underline{f} - a_1 \underline{b}_1 - a_2 \underline{b}_2, \underline{b}_1 \rangle = 0$

$\langle \underline{f} - a_1 \underline{b}_1 - a_2 \underline{b}_2, \underline{b}_2 \rangle = 0$

$\underline{f}$, $\underline{b}_1$, $\underline{b}_2$ known vectors!

applying the rules of scalar product:

$$< \underline{f}, \underline{b}_1 > - a_1 < \underline{b}_1, \underline{b}_1 > - a_2 < \underline{b}_2, \underline{b}_1 > = 0$$

$$< \underline{f}, \underline{b}_2 > - a_1 < \underline{b}_1, \underline{b}_2 > - a_2 < \underline{b}_2, \underline{b}_2 > = 0$$

matrix system

$$\underbrace{\begin{bmatrix} < \underline{b}_1, \underline{b}_1 > & < \underline{b}_2, \underline{b}_1 > \\ < \underline{b}_1, \underline{b}_2 > & < \underline{b}_2, \underline{b}_2 > \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} < \underline{f}, \underline{b}_1 > \\ < \underline{f}, \underline{b}_2 > \end{bmatrix}}_{\underline{b}}$$

How apply to Fourier series?

Együtthatók: $a_0, a_1, a_2, \dots a_N, b_1, b_2, \dots, b_N$

Bázis: $b_0 = 1$; $b_{ci} = \cos(i \frac{2\pi}{T} t)$; $b_{si} = \sin(i \frac{2\pi}{T} t)$

$$\begin{bmatrix} < b_0, b_0 > & < b_{c1}, b_0 > \dots < b_{cN}, b_0 > & < b_{s1}, b_0 > \dots < b_{sN}, b_0 > \\ < b_0, b_{c1} > & < b_{c1}, b_{c1} > \dots < b_{cN}, b_{c1} > & \\ \vdots & & \\ < b_0, b_{sN} > & & \end{bmatrix} \quad \underline{\underline{A}}$$

$$\underline{x}^T = [\, a_0 \;\; a_1 \dots \; a_N \;\; b_1 \; \dots \; b_N \,]$$

$$\underline{b}^T = [\, < f, b_0 > \; < f, b_{c1} > \; \dots \; < f, b_{sN} > \,]$$

scalar product in case of functions

$$< f, g > = \int_0^T f(t) \cdot g(t) \, dt \quad \text{DEFINITION}$$

Fourier series has orthogonal basis:

$$\int_0^T \cos\left(i\frac{2\pi}{T}t\right)\cos\left(j\frac{2\pi}{T}t\right)dt = \langle bc_i, bc_j\rangle = \begin{cases} 0 & i \neq j \\ \frac{T}{2} & i = j \neq 1 \end{cases}$$

$$\int_0^T \sin\left(i\frac{2\pi}{T}t\right)\sin\left(j\frac{2\pi}{T}t\right)dt = \langle bs_i, bs_j\rangle = \begin{cases} 0 & i \neq j \\ \frac{T}{2} & i = j \end{cases}$$

$$\int_0^T \sin\left(i\frac{2\pi}{T}t\right)\cos\left(j\frac{2\pi}{T}t\right)dt = \langle bs_i, bc_j\rangle = 0$$

$$\int_0^T 1\cdot 1\ dt = \langle b_0, b_0\rangle = T$$

$$\begin{bmatrix} \underset{(T)}{\langle b_0, b_0\rangle} & 0 & 0 & & \\ 0 & \underset{(T/2)}{\langle bc_1, bc_1\rangle} & 0 & & \\ 0 & 0 & \underset{(T/2)}{\langle bc_2, bc_2\rangle} & & \\ \vdots & & & \ddots & \\ 0 & & & & \underset{(T/2)}{\langle bs_N, bs_N\rangle} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} \langle f, b_0\rangle \\ \langle f, bc_1\rangle \\ \langle f, bc_2\rangle \\ \\ \langle f, bs_N\rangle \end{bmatrix}$$

$$a_0 = \frac{\langle f, b_0\rangle}{\langle b_0, b_0\rangle} = \frac{1}{T}\int_0^T f\cdot 1\ dt$$

$$a_i = \frac{\langle f, bc_i\rangle}{\langle bc_i, bc_i\rangle} = \frac{2}{T}\int_0^T f\cdot bc_i\ dt = \frac{2}{T}\int_0^T f\cdot\cos\left(i\frac{2\pi}{T}t\right)dt$$

$$b_i = \frac{\langle f, bs_i\rangle}{\langle bs_i, bs_i\rangle} = \frac{2}{T}\int_0^T f\cdot\sin\left(i\frac{2\pi}{T}t\right)dt$$

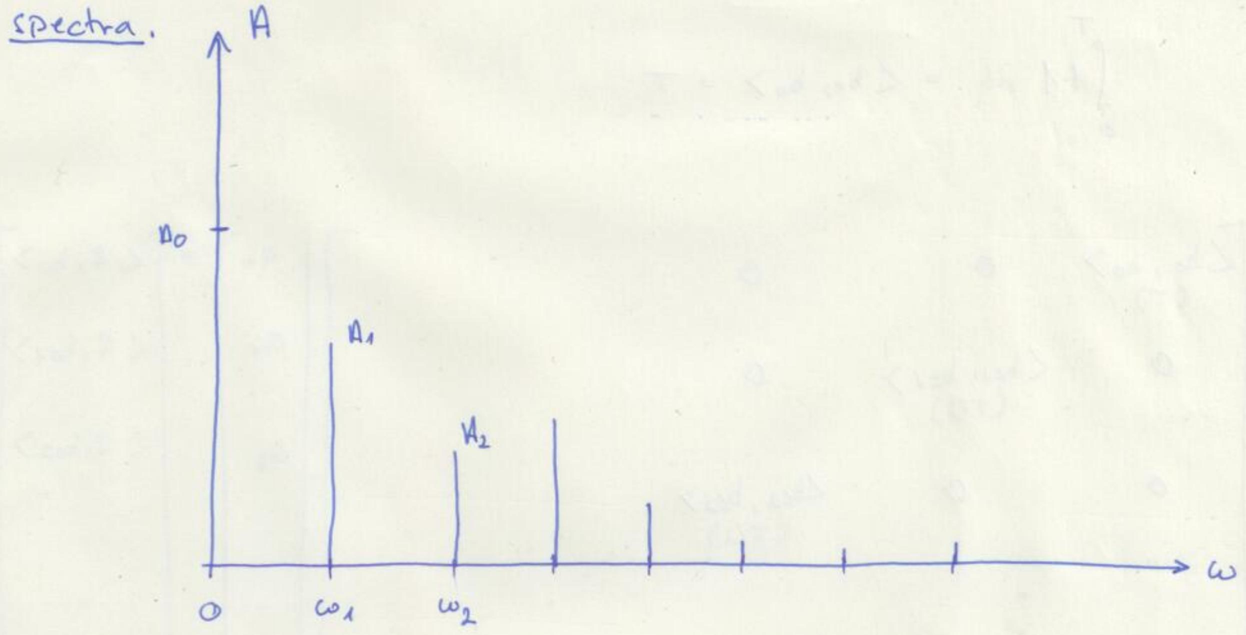
$$f_T(t) = a_0 + \sum_{i=1}^{N} a_i \cos(\omega_i t) + b_i \sin(\omega_i t) \; .$$

$$\Big\downarrow \qquad \omega_i = i\frac{2\pi}{T} = i\,\omega$$

$$f_T(t) = a_0 + \sum_{i=1}^{N} A_i \cos(i\omega t + \phi)$$

$$\boxed{\begin{array}{l} A_i = \sqrt{a_i^2 + b_i^2} \\[2ex] \phi = \tan^{-1} \dfrac{b_i}{a_i} \end{array}} \quad \begin{array}{l} \text{amplitude} \\[2ex] \text{phase} \end{array}$$

<u>spectra.</u>



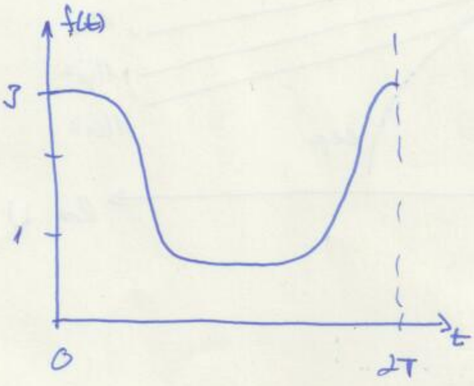
$$\omega = \Delta\omega = \frac{2\pi}{T}$$

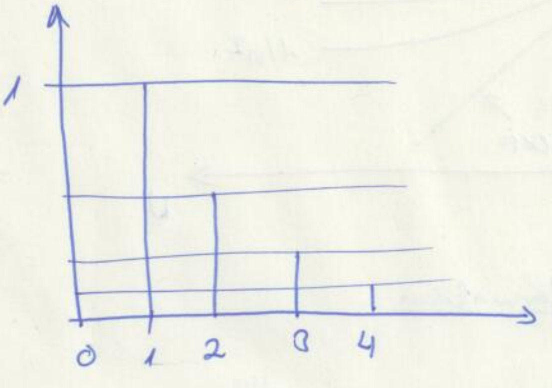
$$\Delta f = \frac{1}{T}$$

Példák: $f(t) = \frac{3}{5-4\cdot\cos(x)}$; $0 \le t < T$;

①

C^{∞} infinite many times differentiable (smooth function)





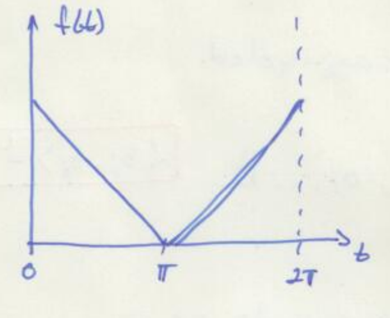
$A_i \cong 2\cdot e^{-0.693 n}$

In general: $\left.\begin{matrix}a_i\\ b_i\end{matrix}\right\} \sim O(\exp(-q\cdot n^r))$; $q = 0.693$

$r = 1$

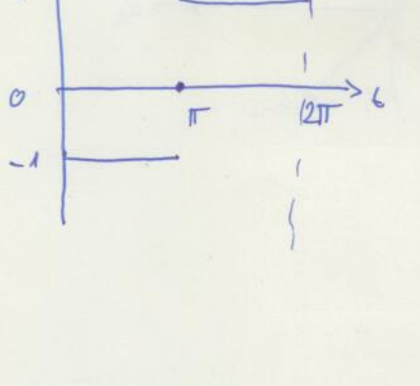
(exponential convergence)

② $f(t) = \text{abs}(x-\pi)$



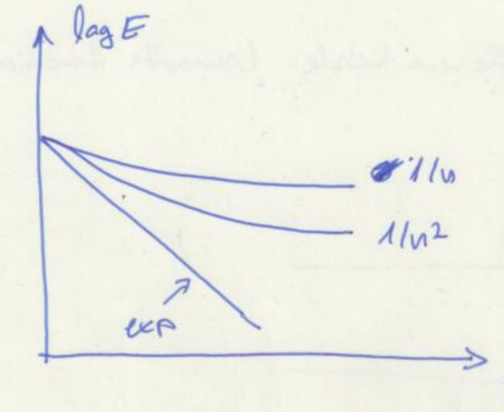
$\left.\begin{matrix}a_i\\ b_i\end{matrix}\right. \sim O(1/n^2)$

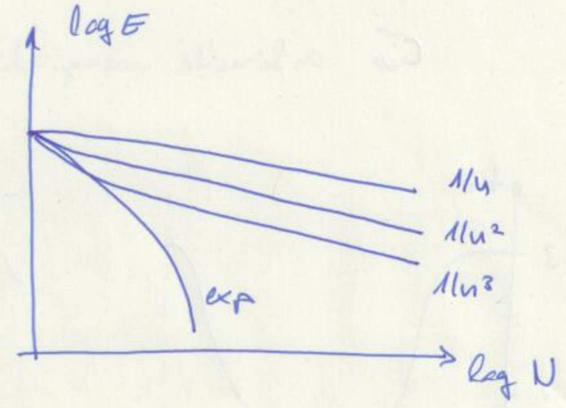
③ $f(t) = \text{sgn}(x-\pi)$



$\left.\begin{matrix}a_i\\ b_i\end{matrix}\right. \sim O(1/n)$

Convergence:



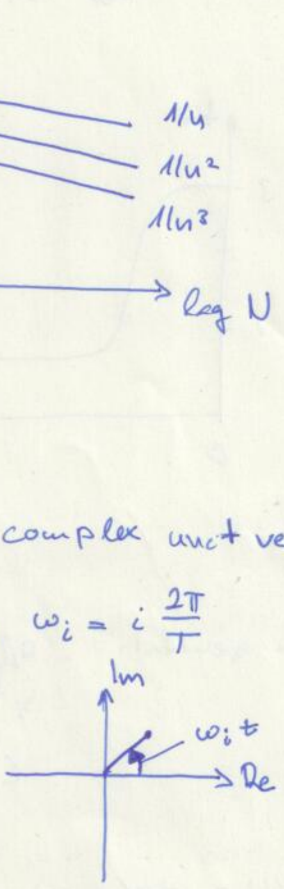


Complex formalism

f(t) $e^{j\omega} = \cos\omega + j\sin\omega$ j - complex unit vector

Let us write the basis as: $\boxed{b_i = e^{j\omega_i t}}$; $\omega_i = i\frac{2\pi}{T}$

$$\underset{T}{f}(t) = \sum_{i=-N}^{N} c_i\, b_i(t)$$



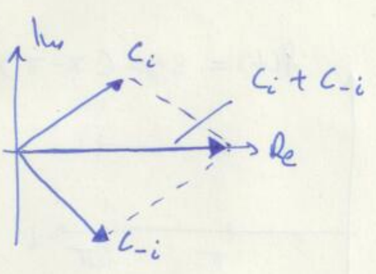
Inner product: $\langle f, g\rangle = \int_0^T f\cdot\underset{\uparrow}{\bar{g}}\,dt$

complex conjugeted.

$$\boxed{c_i = \frac{\langle f, b_i\rangle}{\langle b_i, b_i\rangle}} \quad ; i = -N \ldots, 0, \ldots N \qquad \boxed{\langle b_i, b_i\rangle = T}$$

If f(t) is real, then c_i are complex conjugate pairs.

$$\boxed{c_i = \bar{c}_{-i}}$$



The original coefficients

$$\boxed{\begin{aligned} a_i &= c_i + c_{-i} \\ b_i &= j(c_i - c_{-i}) \end{aligned}} \rightarrow \begin{matrix} A_i \\ \phi_i \end{matrix}$$

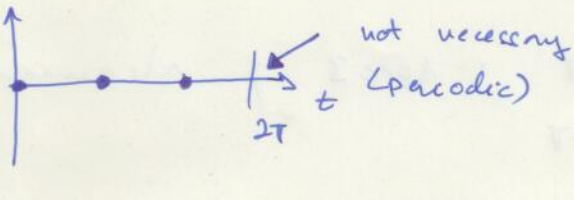
Discrete Fourier Transform (DFT)

Total points: $2N+1$, Time interval: $0 \le t \le T$

$\Delta t = \frac{T}{2N+1}$; $\boxed{t_i = i \cdot \frac{T}{2N+1} = i \cdot \Delta t}$ $i = 0 \dots 2N$

E.G.: $N=1$, $i = 0,1,2$; $\Delta T = \frac{2\pi}{3}$
$T = 2\pi$

$t_1 = 0$
$t_2 = \frac{2\pi}{3}$
$t_3 = \frac{4\pi}{3}$
. . . .



The function: $f(t)$ continuous

$\underline{f} = f(t_i)$ discrete version, (sampled data)

Exponential basis: $\underline{b}_k = e^{j \omega_k t_i}$; $\omega_k = k \frac{2\pi}{T}$; komplex értékű vektor (complex valued vector)

Inner product: $\langle \underline{f}, \underline{g} \rangle = \sum_{i=0}^{2N} f(t_i) \cdot \bar{g}(t_i)$

$C_k = \frac{\langle \underline{f}, \underline{b}_k \rangle}{\langle \underline{b}_k, \underline{b}_k \rangle}$; $\langle \underline{b}_k, \underline{b}_k \rangle = 2N+1$
(Matlab: do not make the division!)

$\boxed{C_k = \overline{C_{-k}}}$ & complex conjugate.

$\downarrow$

$\left.\begin{matrix} a_i \\ b_i \end{matrix}\right\}$ $\begin{matrix} A_i \\ \phi_i \end{matrix}$ (Spectra)

a) $f(t) = \frac{3}{5-4\cdot\cos(t)}$; smooth

b) $f(t) = \text{abs}(-x+\pi)$; abs

c) $f(t) = \text{sign}(x-\pi)$; discont ; discont long

d $f(t) = 3\cdot\cos(x) + 5\sin(5x)$; sepc (hint for aliasing)

e) $f(t) = 1\cdot\sin(9x)$; spec2 ; explanation of aliasing.

f) $f(t) = \begin{cases} \sin(x), & x < 2T \\ \sin(5x), & x > 2\pi \end{cases}$; spec3 ; shannon 3-5

g) $f(t) = (1+0.2\sin(0.2x+2))\cdot\sin(3x)$; ampl-mod

h) $f(t) = \sin((3+0.\overset{01}{\cancel{2}}\sin(0.2\cdot x))\cdot x)$; freq_mod_small

i) 0.2 freq_mod_large

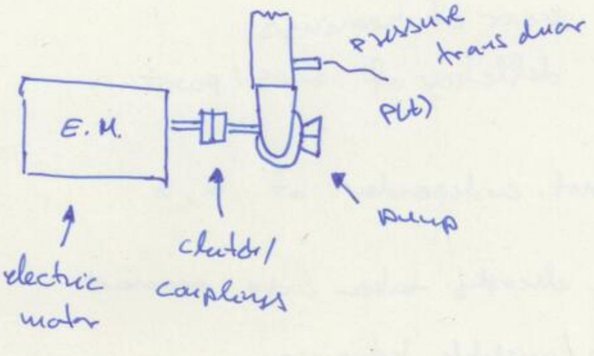
j) $\boxed{\text{chaotic}}$ frekvencia felbontás: $\boxed{\Delta f = \frac{1}{T}}$

(Mikor állítod le!)

k) $\boxed{\text{nonlin}}$ chaoti, de egyéb kam exp( )

Előtte a sima példa!

Vibration Monitoring



Aims: reveal peaks in the spectra and associate physical phenomena

- Hydrodinamic effects (pump)
- Mechanical effects (clutch, others like bearings)
- Motor instability

p(t) → Spectra → Vib. Monit.

- Variable: p(t) → Spectra
- Parameters:

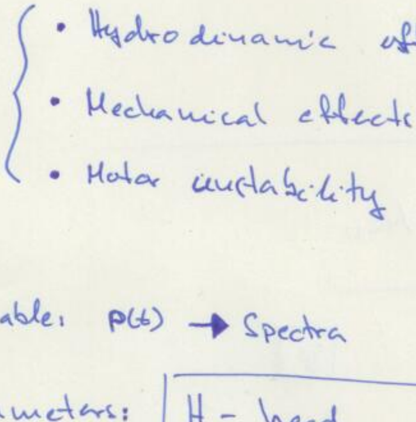
H - head
Q - volume flow rate
n - revolution number
P_i - input power of pump

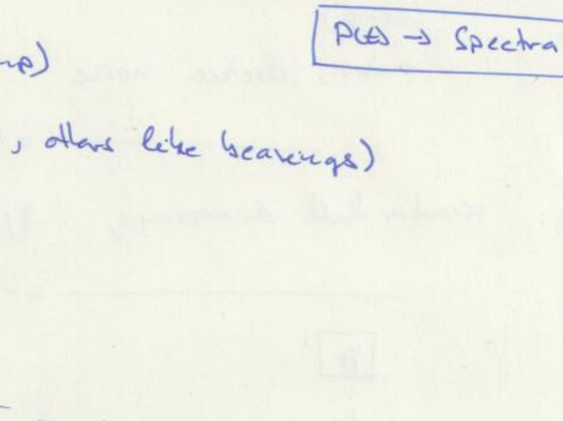
hydrodin. parameters.

characteristic curves

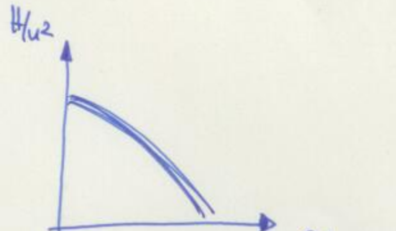
control param: Q, n

$$\eta = \frac{Q \cdot \rho \cdot g \cdot H}{P_i}$$





+ Affinitás:
(Affinity)



every param. is controlled!

- Geometry of impeller / housing
- Number of blades

Mechanical parameters: eccentricity (geometric prop)
imbalance (mass direction of inertia)
aligment/ fitting
wear of bearings
deflection of axis / pivot

Electric motor: $\left.\begin{matrix} I \\ U \end{matrix}\right\}$ not independent of Q, n

n - already taken into account

stable/unstable behaviour

Egyéb: hőmérséklet T
(temp)
outer, electric noise

———— o ————

Strategy: water fall diagrams, f/f_r ; Q/Q_N

———— o ————

$\boxed{A}$

$\boxed{B}$

n=630 1/min excentr/imb
+
nonlin.

Signal Processing

• Static characteristics of a signal

• Average or mean value:

$$\bar{y} = \frac{\int_{t_1}^{t_2} y(t)\,dt}{\int_{t_1}^{t_2} dt} = \frac{1}{t_2-t_1}\cdot\int_{t_1}^{t_2} y(t)\,dt \qquad \text{continuous}$$

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i \qquad \text{discrete}$$

• Root-mean-square RMS:

Example 4

Electric power dissipation: $P = I^2 R$

$$\text{Total: } \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} I^2(t)\cdot R\,dt$$

$$\text{if } R \text{ is constant: } P_{tot} = R\int_{t_1}^{t_2} I^2(t)\,dt$$

$$\text{equivalent } I_e : P_{tot} = I_e^2\cdot(t_2-t_1)\cdot R$$

$$\left.\begin{matrix} P_{tot} = R\int_{t_1}^{t_2} I^2(t)\,dt \\ P_{tot} = I_e^2\cdot(t_2-t_1)\cdot R \end{matrix}\right\} I_e = \sqrt{\frac{1}{t_2-t_1}\cdot\int_{t_1}^{t_2} I^2\,dt}$$

RMS

$$\downarrow$$

$$y_{rms} = \sqrt{\frac{1}{t_2-t_1}\cdot\int_{t_1}^{t_2} y^2(t)\,dt} \qquad \text{continuous}$$

$$y_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} y_i^2} \qquad \text{discrete}$$

· Correlation

- Linear relationship between two datasets!

$$x_i = x_1, \dots, x_N \qquad t_i = 1, \dots, N$$

$$y_i = y_1, \dots, y_N$$

$$R(x,y) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{S_x \, S_y} \quad ; \quad S_x^2 = \sum_{i=1}^{N} (x - \bar{x})^2 \qquad S_y^2 = \sum_{i=1}^{N} (y - \bar{y})^2$$

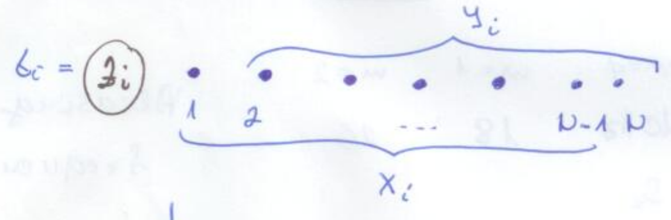
$$\boxed{-1 \leq R \leq 1}$$

$|R| \approx 1$ -> Strong linear relationship

- Autocorrelation.

$\left.\begin{matrix} x_i \\ y_i \end{matrix}\right\}$ the set of data points $\quad t_i = 1, \dots, N$

Correlation with itself.



$$x_i = z_i \qquad i = 1, \dots, N-1$$

$$\boxed{z_{i+1} = y_i} \qquad i = 1, \dots, N-1$$

$\downarrow$

First order ~~ca~~ auto correlation.

$$R_1^A(\cancel{z}\,x) = \frac{\sum_{i=1}^{N-1} (x_i - \bar{x}_{(1)})(x_{i+1} - \bar{x}_{(2)})}{\sqrt{\sum_{i=1}^{N-1} (x_i - \bar{x}_{(1)})^2 \cdot \sum_{i=1}^{N} (\cancel{y}\,x_{i+1} - \cancel{\bar{y}}\,\bar{x}_{(2)})^2}}$$

$$\boxed{\begin{aligned} \bar{x}_{(1)} &= \frac{1}{N-1} \sum_{i=1}^{N-1} x_i \\ \bar{x}_{(2)} &= \frac{1}{N-1} \sum_{i=1}^{N-1} x_{i+1} \end{aligned}}$$

Simplifications.

For long signals. $\bar{x}_{(1)} \cong \bar{x}_{(2)} \cong \bar{x}$

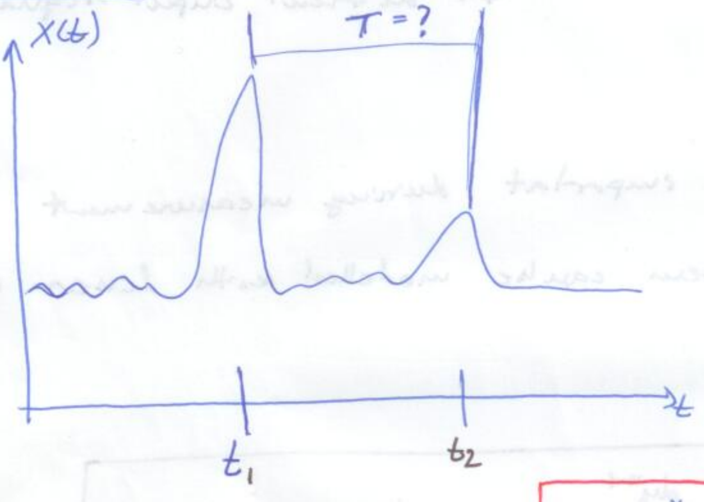
$$\sum_{i=1}^{N-1} (x_i - \bar{x}_{(1)})^2 \cong \sum_{i=1}^{N-1} (x_{i+1} - \bar{x}_{(2)})^2 \cong \sum_{i=1}^{N} (x_i - \bar{x})^2$$

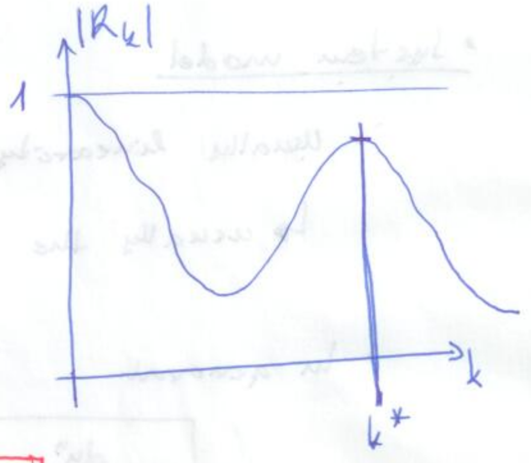
$$\boxed{R_1^A(x) \cong \frac{\sum_{i=1}^{N-1}(x_i-\overline{x})(x_{i+1}-\overline{x})}{\sum_{i=1}^{N}(x_i-\overline{x})^2}}$$

k - order autocorrelation:

$$R_k^A(x) \cong \frac{\sum_{i=1}^{N-k}(x_i-\overline{x})(x_{i+k}-\overline{x})}{\sum_{i=1}^{N}(x_i-\overline{x})^2}$$ shifted k times!

Application





$$t_2 - t_1 = \boxed{T = k^* \cdot \Delta t}$$

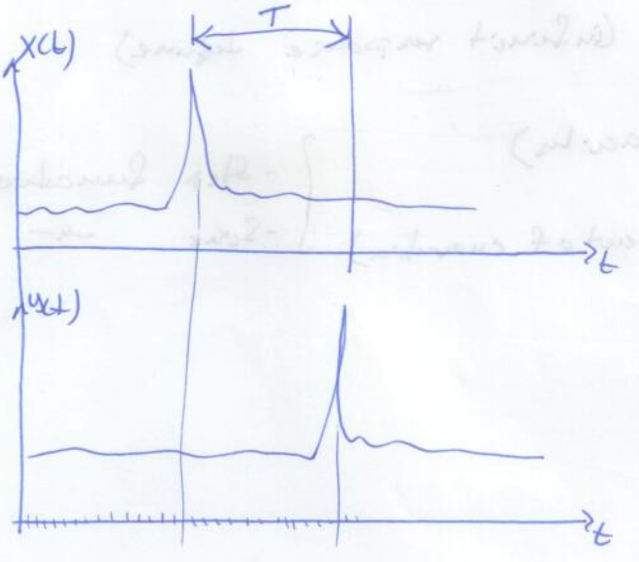
- Cross - correlation

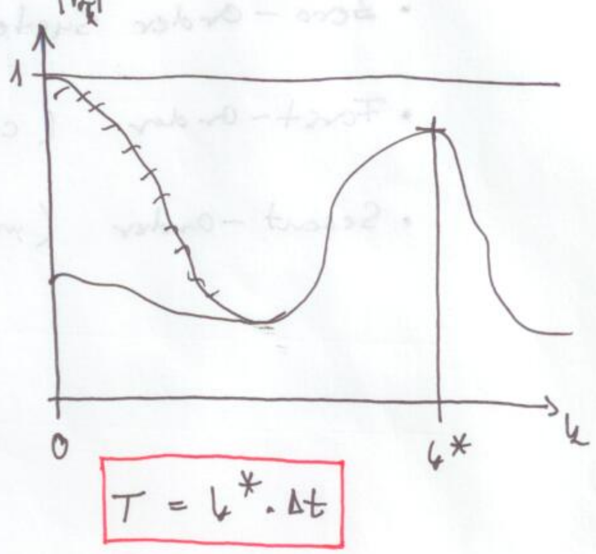
x_i, y_i } distinct

$$R_k^C(x,y) \cong \frac{\sum_{i=1}^{N-k}(x_i-\overline{x})(y_{i+k}-\overline{y})}{\sqrt{\sum_{i=1}^{N}(x_i-\overline{x})^2 \cdot \sum_{i=1}^{N}(y_i-\overline{y})^2}}$$

$t_i = t_1, \dots, t_N$

usually: $\Delta t = \text{const}$





$$\boxed{T = k^* \cdot \Delta t}$$