• Basic principles (Conservation laws)

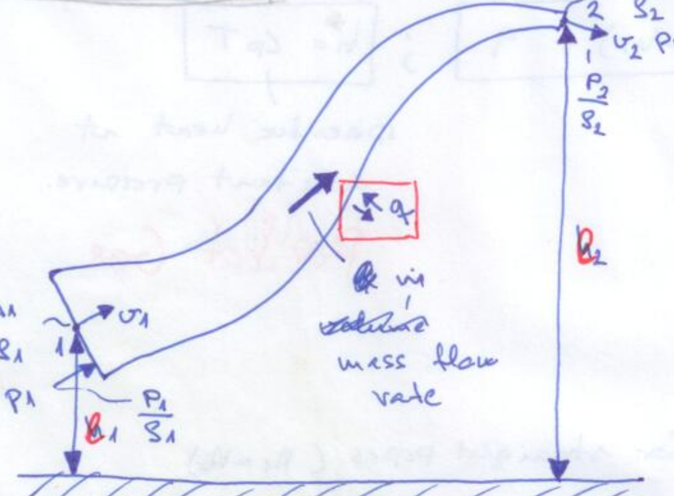
Stationary

- Mass
- Momentum (mechanical energy)
- Energy

in one piece

— c —

• Pipe (system)



• Mass

$\dot{m}_{in} = A_1 \cdot v_1 \cdot \rho_1 \quad [kg/s]$

$\dot{m}_{out} = A_2 v_2 \rho_2 \quad [kg/s]$

$$\boxed{A_1 \cdot v_1 \cdot \rho_1 = A_2 \cdot v_2 \cdot \rho_2} \quad (1)$$

Continuity

v_1, v_2 – mean velocities

• **Momentum** (Newton 2nd Law) $F = m \cdot a$

equivalent statement

$\Delta p = \Delta E_m$

$\frac{\text{Change of Force}}{\text{Unit Area}}$ = Change of Mechanical Energy.

end state
initial state

$$\boxed{\boxed{p_1 - p_2} = \frac{\rho_1}{2}(v_2^2 - v_1^2) + \rho_1 g (h_2 - h_1)} \quad (2) \; + \boxed{\Delta p'}$$

kinetic energy → $\boxed{\frac{1}{2}(\rho_2 v_2^2 - \rho_1 v_1^2)}$

potential energy → $\boxed{g(\rho_2 h_2 - \rho_1 h_1)}$

$\Delta p'$ → friction loss

Bernoulli equation.

• **Energy** ~~Total~~

Change of Total Energy = Heat loss / flow + Mechanical Work.

$$\underbrace{(u_2 - u_1)}_{\text{internal energy}} + \underbrace{\frac{1}{2}(v_2^2 - v_1^2) + g(h_2 - h_1)}_{\text{mechanical energy}} = \underbrace{q}_{\text{heat flow}} \quad \left[\frac{J}{kg}\right] + \underbrace{\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)}_{\text{mechanical work}}$$

important

specified to mass flow rate

$$h^* = u + \frac{p}{\rho} \rightarrow u = h^* - \frac{p}{\rho}$$

h^* ↓ enthalpy

$$(h_2^* - h_1^*) - \frac{1}{\rho}(P_2 - P_1) + \frac{v_2^2 - v_1^2}{2} + g(h_2 - h_1) = q + \frac{P_1}{\rho_1} \cancel{\ } - \frac{P_2}{\rho_2}$$

II. Law of Thermodynamics

$$\boxed{(h_2^* - h_1^*) + \frac{1}{2}(v_2^2 - v_1^2) + g(h_2 - h_1) = q} \quad ; \quad \boxed{h^* = c_p T}$$

c_p: specific heat at constant pressure.

~~Perfect~~ Ideal Gas

- Water distribution systems
  - $\rho_1 = \rho_2 = \rho$
  - Neglecting the thermal effect.

Continuity: $\boxed{A_1 \cdot v_1 = A_2 \cdot v_2}$ for straight pipes: ($A_1 = A_2$)

$$\boxed{v_1 = v_2}$$

Bernoulli equation:

$$\boxed{P_1 - P_2 = \frac{\rho}{2}(v_2^2 - v_1^2) + \rho \cdot g\,(h_2 - h_1) + \Delta p'}$$

$\Delta p'$: pressure loss

in terms of pressure

$/ \rho \cdot g$

$$\boxed{\frac{P_1 - P_2}{\rho \cdot g} = \frac{v_2^2 - v_1^2}{2 \cdot g} + (h_2 - h_1) + \Delta h'}$$ in terms of height

$\frac{P_1 - P_2}{\rho \cdot g}$ ↓ head ↓ $H_1 - H_2$

$\Delta h'$ ↓ head loss

- Pressure losses / Head losses

- Straight pipes

$$\boxed{\Delta h' = \lambda \cdot \frac{L}{D} \frac{v^2}{2g}}$$

λ - friction coefficient

L - pipe lenght

D - pipe inner diameter

λ - depend on - the Reynolds number $Re = \frac{v \cdot D}{\nu}$

$$\boxed{\lambda = f\left(Re, \frac{e}{D}\right)}$$

ν - kinematic viscosity $[m^2/s]$

- relative roughness $\frac{e}{D}$

e - roughness [mm], [m]

- For laminar flow $\boxed{Re < 2300 : \lambda = 64/Re}$
- Turbulent smooth flow $\boxed{4000 < Re < 10^5 : \lambda \cong 0.316/\sqrt[4]{Re}}$
- Otherwise: Darcy - Weisbach equation / Colebrook - White equation

- In terms of flow rate $Q = v \cdot A = v \cdot \frac{D^2 \pi}{4} \Rightarrow \boxed{v = \frac{4Q}{D^2 \pi}}$

$$\boxed{\Delta h' = \lambda \frac{8 \cdot L}{g D^5 \pi^2} \cdot Q^2 = k Q^2} \quad \boxed{\Delta h' \sim Q^2}$$

- other elements

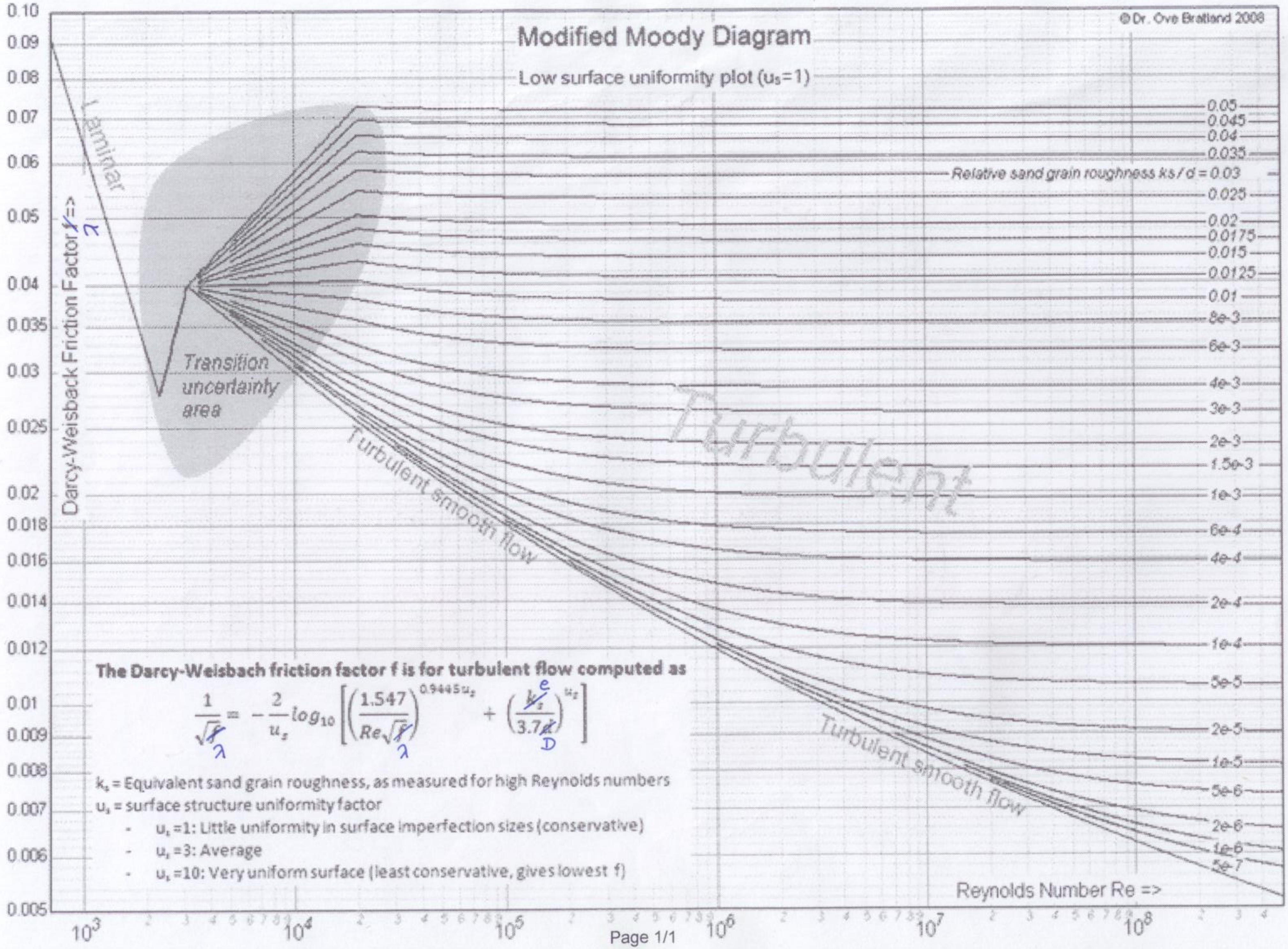
$$\boxed{\Delta h' = \zeta \cancel{\frac{8}{D \cdot g^2}} \frac{v^2}{2g} = \zeta \frac{8}{g D^4 \pi^2} \cdot Q^2 = k \cdot Q^2} \quad \boxed{\Delta h' \sim Q^2}$$

ζ - loss factor

- Estimated from Tables and Handbooks
- From measurement.

- Elbows
- Values
- ~~Chokings~~ throtles
- Confusers
- Diffusers

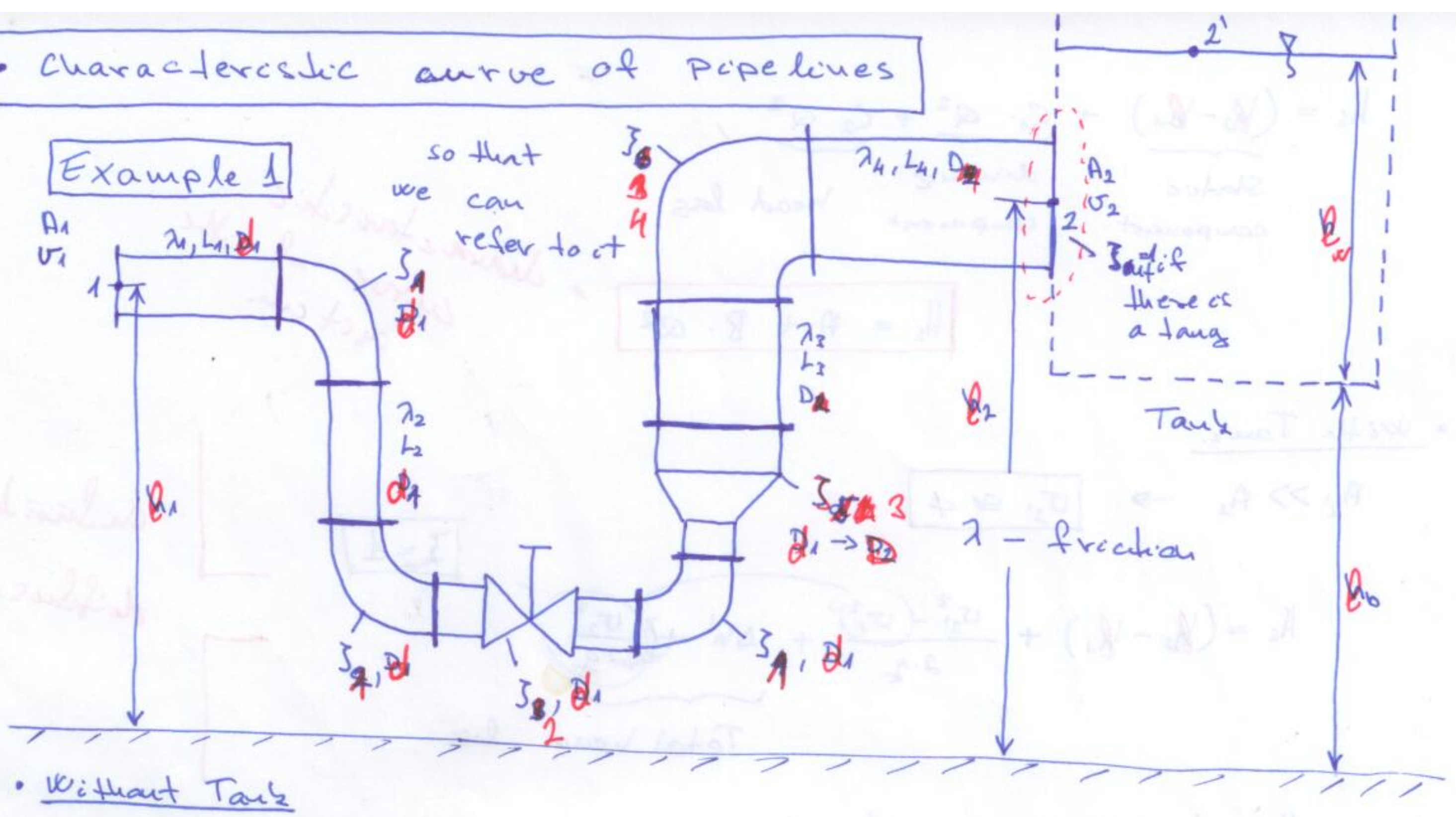
⋮



© Dr. Ove Bratland 2008
Modified Moody Diagram
Low surface uniformity plot (u_s=1)
Darcy-Weisback Friction Factor ~~f~~ λ =>
Laminar
Transition uncertainty area
Turbulent
Turbulent smooth flow
Turbulent smooth flow
Relative sand grain roughness ks / d = 0.03
0.05
0.045
0.04
0.035
0.025
0.02
0.0175
0.015
0.0125
0.01
8e-3
6e-3
4e-3
3e-3
2e-3
1.5e-3
1e-3
6e-4
4e-4
2e-4
1e-4
5e-5
2e-5
1e-5
5e-6
2e-6
1e-6
5e-7
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.035
0.03
0.025
0.02
0.018
0.016
0.014
0.012
0.01
0.009
0.008
0.007
0.006
0.005
10^3
10^4
10^5
10^6
10^7
10^8
Reynolds Number Re =>
The Darcy-Weisbach friction factor f is for turbulent flow computed as
$$\frac{1}{\sqrt{\cancel{f}\,\lambda}} = -\frac{2}{u_s}\log_{10}\left[\left(\frac{1.547}{Re\sqrt{\cancel{f}\,\lambda}}\right)^{0.9445u_s} + \left(\frac{\cancel{k_s}\,e}{3.7\cancel{d}\,D}\right)^{u_s}\right]$$
k_s = Equivalent sand grain roughness, as measured for high Reynolds numbers
u_s = surface structure uniformity factor
- u_s =1: Little uniformity in surface imperfection sizes (conservative)
- u_s =3: Average
- u_s =10: Very uniform surface (least conservative, gives lowest f)

• Characteristic curve of pipelines

Example 1



• Without Tank

Bernoulli equation between 1 and 2.

$$\boxed{H_s = H_1 - H_2 = \frac{v_2^2 - v_1^2}{2 \cdot g} + (h_2 - h_1) + \Delta h'}$$

$$v_1 = \frac{4Q}{D_1^2 \pi} \qquad v_2 = \frac{4Q}{D_2^2 \pi}$$

$$H_s = (h_2 - h_1) + \frac{8}{g D_2^4 \pi^2} Q^2 - \frac{8}{g D_1^4 \pi^2} Q^2 + \Delta h'$$

$$\left(\frac{8}{g D_2^4 \pi^2} - \frac{8}{g D_1^4 \pi^2}\right) \cdot Q^2 = \underline{\underline{C_1 \cdot Q^2}}$$

kinetic en.

$$\Delta h' = \lambda_1 \frac{L_1 \cdot 8}{g d^5 \pi^2} \cdot Q^2 + \zeta_1 \frac{8}{g d^4 \pi^2} \cdot Q^2 + \lambda_2 \frac{8 L_2}{g d^5 \pi^2} \cdot Q^2 + \zeta_1 \frac{8}{g d^4 \pi^2} \cdot Q^2$$

$$+ \zeta_2 \frac{8}{g d^4 \pi^2} Q^2 + \zeta_1 \frac{8}{g d^4 \pi^2} Q^2 + \zeta_3 \frac{8}{g d^4 \pi^2} Q^2 + \lambda_3 \frac{8 L_3}{g D_2^5 \pi^2} Q^2$$

$$+ \zeta_4 \frac{8}{g D_2^4 \pi^2} Q^2 + \lambda_4 \frac{8 L_4}{g D_2^5 \pi^2} \cdot Q^2 =$$

$$\boxed{\begin{aligned} \Delta h' &= \frac{8}{g d^4 \pi^2}\left(\lambda_1 \frac{L_1}{d} + \zeta_1 + \lambda_2 \frac{L_2}{d} + \zeta_1 + \zeta_2 + \zeta_1 + \zeta_3\right) \cdot Q^2 + \\ &\quad \frac{8}{g D_2^4 \pi^2}\left(\lambda_3 \frac{L_3}{D_2} + \zeta_4 + \lambda_4 \frac{L_4}{D_2}\right) \cdot Q^2 = \underline{\underline{C_2 \cdot Q^2}} \end{aligned}}$$

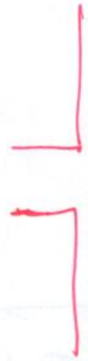
$$H_s = \underbrace{(z_2 - z_1)}_{\text{Static component}} + \underbrace{C_1 \cdot Q^2}_{\text{dinamic component}} + \underbrace{C_2\, Q^2}_{\text{head loss}}$$

$$\boxed{H_s = A + B \cdot Q^2}$$ ← characteristic curve of the system.

- With Tank.

$$A_{2'} >> A_2 \rightarrow \boxed{v_{2'} \cong 0}$$

$$H_s = (z_{2'} - z_1) + \frac{v_{2'}^2 - v_1^2}{2 \cdot g} + \underbrace{\Delta h' + \xi_{out} \frac{v_2^2}{2g}}_{\text{Total head loss}} \qquad \boxed{\xi_{out} = 1}$$



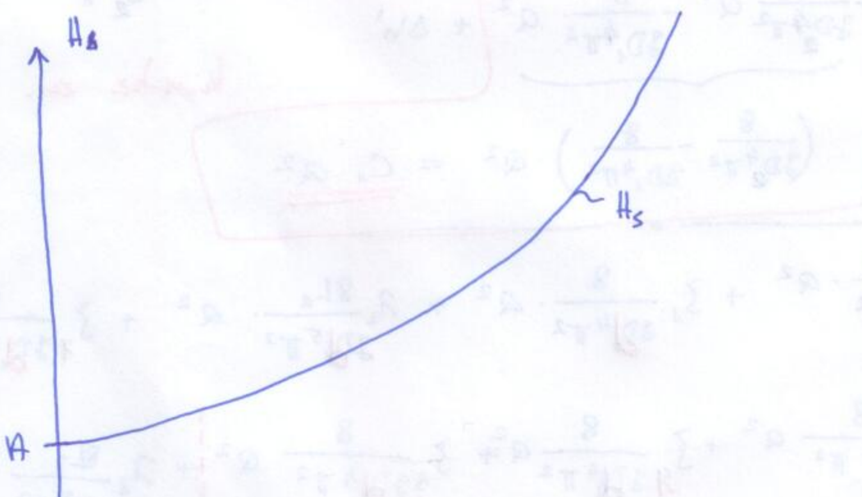
on tank diffuser

$$H_s = (z_0 + z_w - z_1) + \frac{v_2^2 - v_1^2}{2g} + \Delta h'$$

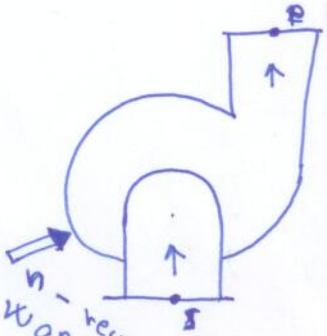
$$\qquad\qquad\qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\qquad\qquad\qquad\quad C_1\, Q^2 \qquad C_2 Q^2$$

$$\boxed{H_s = A + B \cdot Q^2}$$



• Characteristic curves of Pumps



n - revolution number
WORK → Prod. H_p

pump

s - suction side
p - pressure side

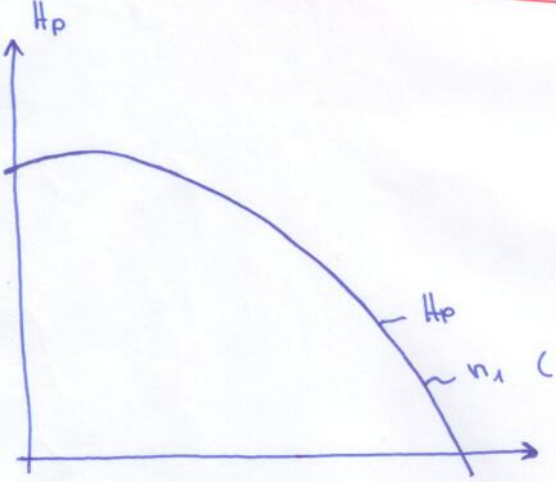
• Bernoulli equation is not valid.
• By definition:

$$\boxed{H_p = \frac{v_p^2 - v_s^2}{2g} + (h_p - h_s) + \frac{p_p - p_s}{\rho \cdot g}}$$

$H_p = f(Q)$ — From Datasheet of the Producer.
— Measurement.

$a < 0$
b
c

$$\boxed{H_p = a \cdot Q^2 + b\,Q + c}$$

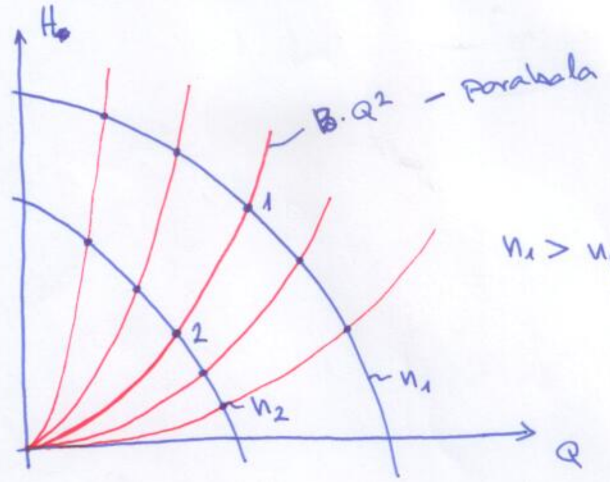


n_1 (for a given revolution number)
P.I.: = 1440 1/min

$\Rightarrow$ $a = f(n)$
$b = f(n)$
$c = f(n)$

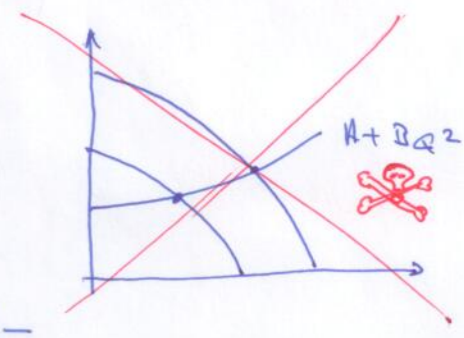
• Changing the revolution number.

AFFINITY



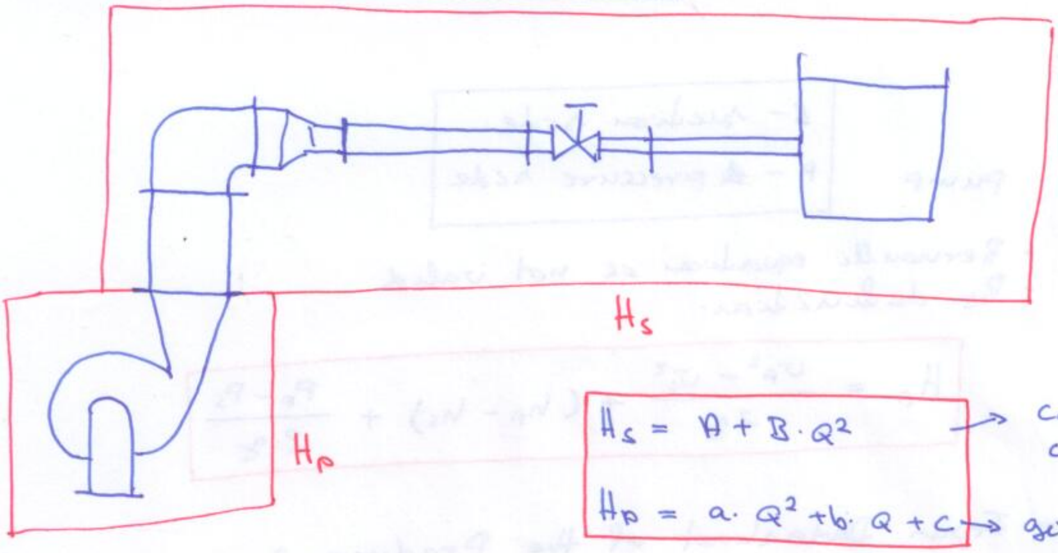
$n_1 > n_2$

$$\boxed{\frac{Q_1}{Q_2} = \frac{n_1}{n_2} \qquad \frac{H_1}{H_2} = \left(\frac{n_1}{n_2}\right)^2}$$



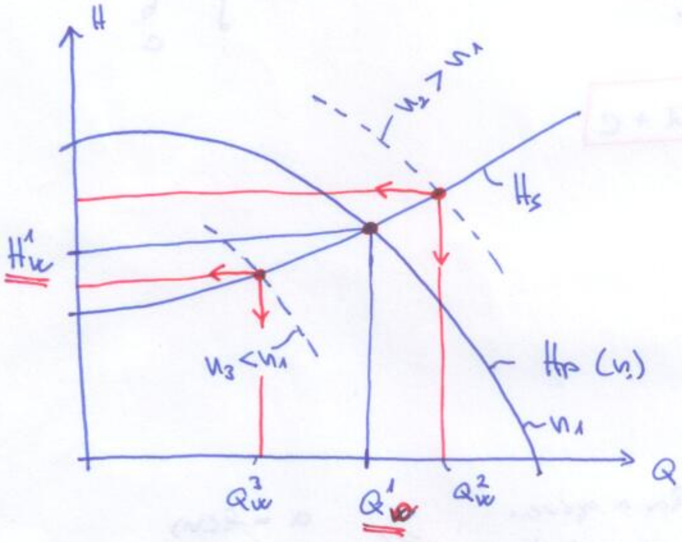
• Operation point

Example 2



$H_s = A + B \cdot Q^2$ → can calculated

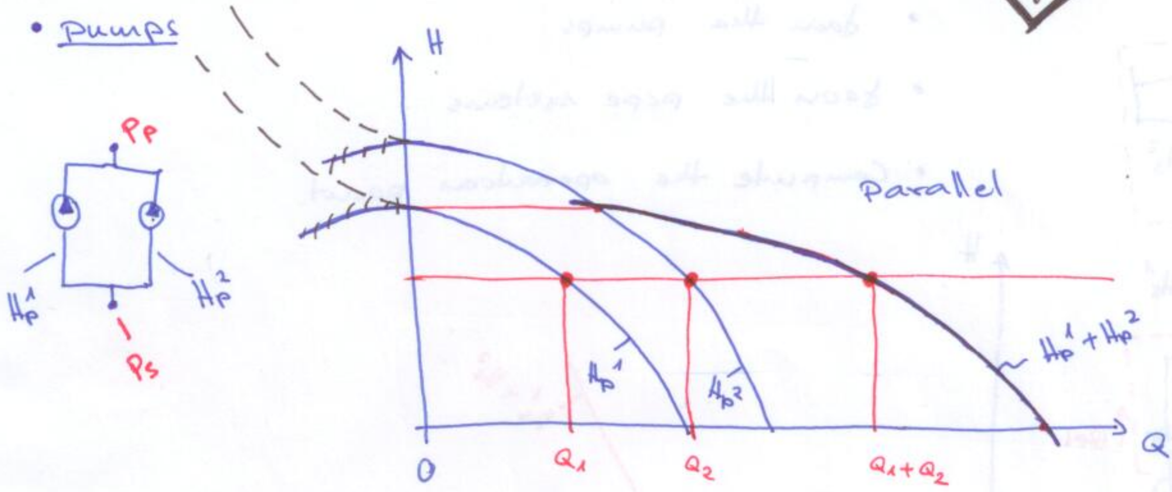
$H_p = a \cdot Q^2 + b \cdot Q + c$ → given by manufacturer

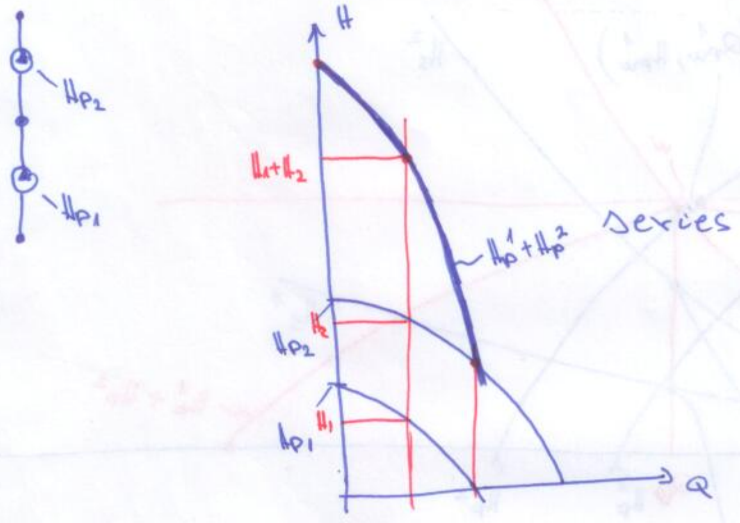


• Parallel and series connections

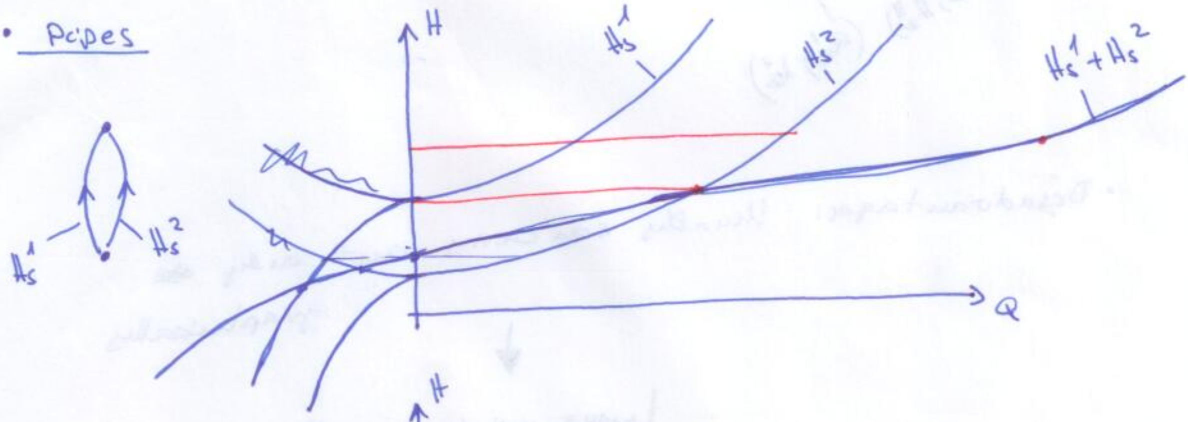
2

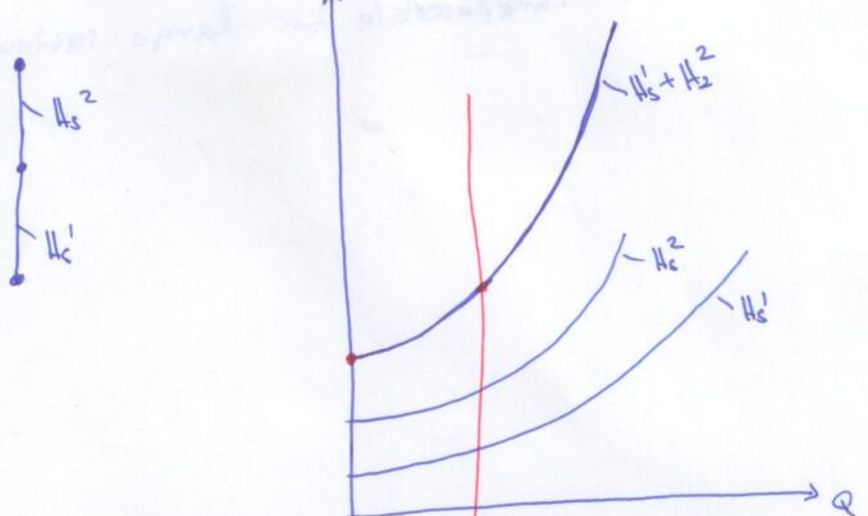
• pumps





• Pipes



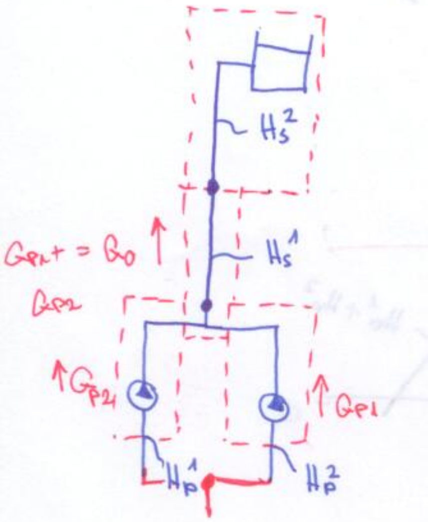


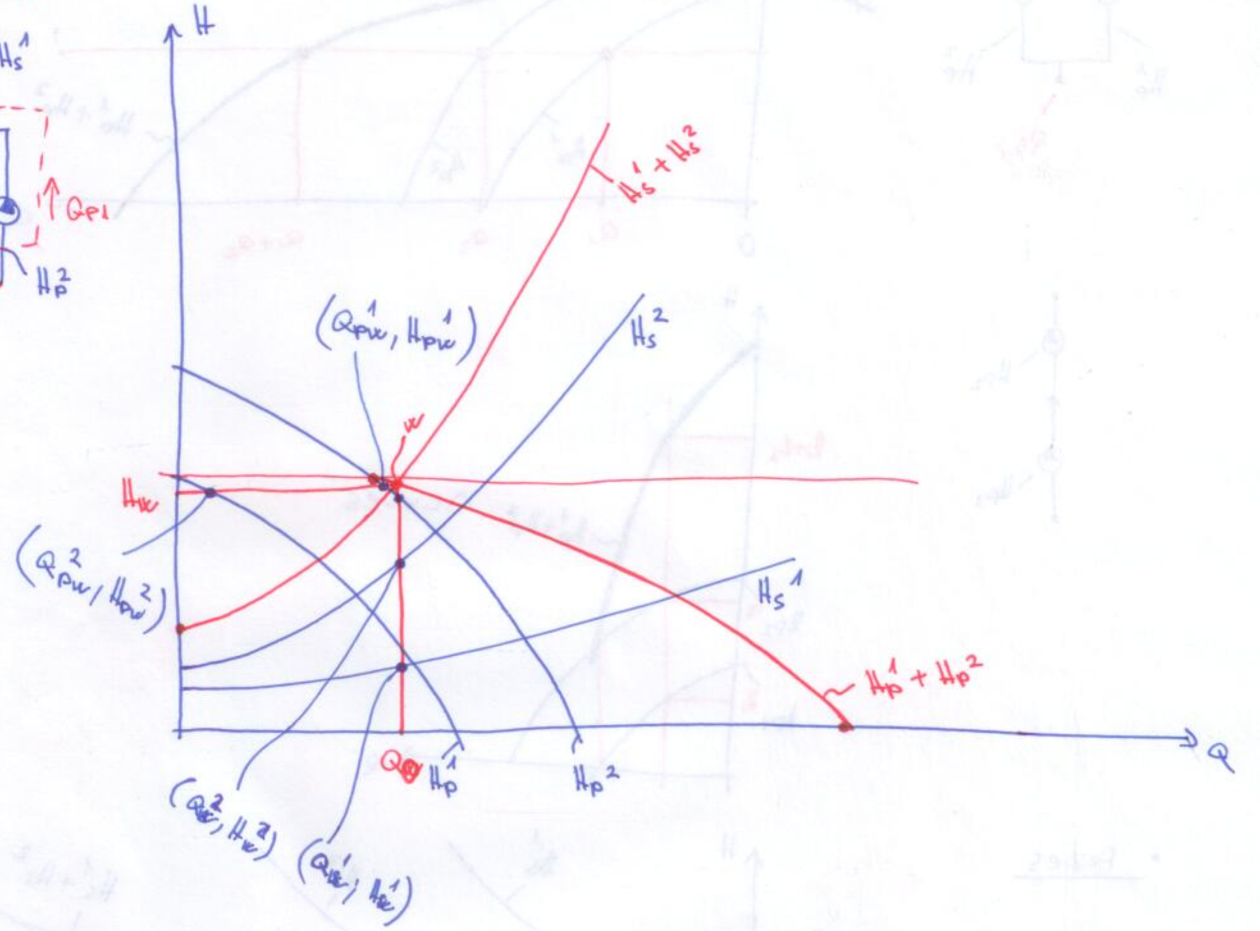
• Simple example <u>Example 3</u>

• Draw the pumps

• Draw the pipe systems

• Compute the operation point.





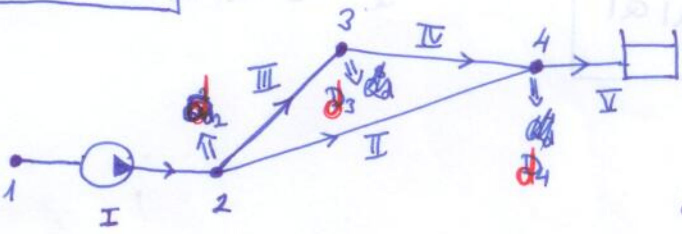
• Disadvantages: Usually can carry out only graphically

↓

Impossible for large systems

• General solution technique for pipe network.

Example 4



tölt – fill
ürít – empty

$d_2 + d_3 + d_4 = 4600\ m^3/h$

- NODES (arabic numbers)
- BRANCHES (roman numbers)
  - straight pipes (λ, D, L, e)
  - valves (ζ, D)
  - throttles (ζ, D)
  - reservoires (h_b, h_r)
  - ⋮

Unknowns:
p_1 Q_I
p_2 Q_{II}
p_3 Q_{III}
p_4 Q_{IV}
Q_V

Σ 9 unknown variables

• Principle equations

• NODES (Continuity equation; Conservation of mass)

$$\sum_{i=1}^{n} \rho_i Q_i = \rho_d \, d \quad \text{- Compressible flow}$$

$$\boxed{\sum_{i=1}^{n} Q_i = d} \quad \text{- Incompressible flow}$$

(This equation is automatically satisfied in Example 3, in graphical solution)

• BRANCHES

– Bernoulli equation (general case)

$$p_1 - p_2 = \frac{\rho}{2}\left(v_2^2 - v_1^2\right) + \rho \cdot g \left(h_2 - h_1\right) + \Delta p'$$

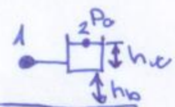
– Specifically,

• Straight pipes:

$Q^2 \rightarrow Q|Q|$

$$\boxed{p_1 - p_2 = \rho \cdot g \left(h_2 - h_1\right) + \lambda \frac{8 \cdot L \cdot \rho}{D^5 \pi^2} Q \cdot |Q|}$$

• reservoires



$$\boxed{p_1 - p_0 = \rho \cdot g \left(h_b + h_w - h_1\right)}$$

- <u>Minor losses</u> (valves, throtles, elbows, ...)

$$\boxed{P_1 - P_2 = \zeta \frac{8\cdot\rho}{D^4\cdot\pi^2}\, Q|Q|}$$

$Q^2 \Rightarrow Q|Q|$

- <u>pumps</u>

By definition

$$H_P(Q) = \frac{v_2^2 - v_1^2}{2\cdot g} + (h_2 - h_1) + \frac{P_2 - P_1}{\rho\cdot g} \qquad /\cdot\rho\cdot g$$

$\uparrow$ given by the manufacturer

$$\rho\cdot g\, H_P = (P_2 - P_1) + \frac{\rho}{2}(v_2^2 - v_1^2) + \rho g (h_2 - h_1)$$

Pressure side $Q = v_2 \cdot A_2$; $A_2 = \frac{D_2^2\pi}{4} \Rightarrow \boxed{v_2 = \frac{4Q}{D_2^2\pi}}$

suction side. $\boxed{v_1 = \frac{4Q}{D_1^2\pi}}$

$$\boxed{\rho\cdot g\, H_P = (P_2 - P_1) + \cancel{\frac{8\cdot\rho}{\pi^2}\left(\frac{1}{D_2^4} - \frac{1}{D_1^4}\right)}\cdot Q^2 + \cancel{\rho\cdot g (h_2 - h_1)}}$$

$\downarrow$ $\downarrow$

neglecting.

• Collecting the equations

• NODES **Eq. Sys. 1**

$$
\boxed{
\begin{aligned}
&2: && Q_I - Q_{II} - Q_{III} - d_2 = \phi \\
&3: && Q_{III} - Q_{IV} - d_3 = \phi \\
&4: && Q_{IV} + Q_{II} - Q_V - d_4 = \phi
\end{aligned}
} \quad 3
$$

• Boundary conditions (If there is no reservoir)

$$\boxed{P_1 = P_0} \quad 1$$

• BRANCHES

I: (pump)

$$H_P = a \cdot Q_I^2 + b\, Q_I + c = f(Q_I)$$

$$\boxed{\rho \cdot g\, H_P = (P_2 - P_1) + \cancel{\frac{8 \cdot \rho}{\pi^2}\left(\frac{1}{D_2^4} - \frac{1}{D_1^4}\right)} \cdot Q_I^2 + \cancel{\rho \cdot g (h_2 - h_1)}} \quad 1$$

neglecting.

(Pipes)

$$
\boxed{
\begin{aligned}
&\text{II.} && P_2 - P_4 = \rho \cdot g (h_4 - h_2) + \lambda_{II} \frac{8\, L_{II}\, \rho}{D_{II}^5 \cdot \pi^2} \cdot Q_{II} \,|Q_{II}| \\
&\text{III:} && P_2 - P_3 = \rho \cdot g (h_3 - h_2) + \lambda_{III} \cdot \frac{8 \cdot L_{III} \cdot \rho}{D_{III}^5 \cdot \pi^2}\, Q_{III} \,|Q_{III}| \\
&\text{IV:} && P_3 - P_4 = \rho \cdot g (h_4 - h_3) + \lambda_{IV} \frac{8 \cdot L_{IV} \cdot \rho}{D_{IV}^5 \cdot \pi^2}\, Q_{IV} \cdot |Q_{IV}|
\end{aligned}
} \quad 3
$$

(reservoir)

$$\text{V:} \quad \boxed{P_4 - P_0 = \rho \cdot g (h_b + h_w - h_4)} \quad 1$$

$$\boxed{\Sigma \;\; 9 \text{ equations}} \checkmark$$

Problem: Nonlinear equation system!!

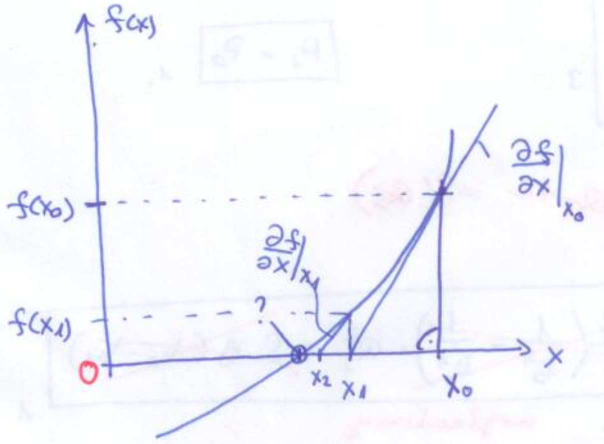
• Solving nonlinear equations

$X^3 = 2 \sin x$

$0 = \underbrace{x^2 - 2\sin x}_{f(x)}$; $f(x) = 0$

• 1 Equation: (Newton's method)

$f(x) = \phi$, where $f(x)$ is a general function.



• zeroth approximation, (initial guess) $\boxed{X_0}$

• first approximation,

$$\left.\frac{\partial f}{\partial x}\right|_{x_0} = \frac{f(x_0)}{x_0 - x_1} \quad ; \quad \left.\frac{\partial f}{\partial x}\right|_{x_0} = f'(x_0)$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

• Second approximation

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

$\vdots$

• General

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Example 5

$f(x) = x^3 - 2 \cdot \sin x$

$x^3 - 2\sin x = \phi$

Initial guess $\underline{\underline{x_0 = 2}}$

no. of steps	x	f	f'
0	2	6.181	12.832
1	1.518	1.503	6.811
2	1.297	0.259	4.512
3	1.240	0.015	3.965
4	1.236	0.00007	3.928
5	1.236	0.00000	3.928

- System of equations (k number of equation)

- Let $\underline{X}$ be the vector of unknowns

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$$

- Let $\underline{f}$ be the vector of nonlinear functions

$$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{bmatrix}$$

For instance.

$$\boxed{\begin{aligned} f_1 &= x_1^3 - 2\sin x_3 \\ f_2 &= x_1 \cdot x_2^2 - 1 \\ &\vdots \\ f_k &= x_1 \cdot x_{10} - x_k^2 \end{aligned}}$$

$$\boxed{\underline{f}(\underline{X}) = \underline{\phi}}$$

$$f_1(x_1, x_2, \dots, x_k) = \phi$$
$$f_2(x_1, x_2, \dots, x_k) = \phi$$
$$\vdots$$
$$f_k(x_1, x_2, \dots, x_k) = \phi$$

For instance:

$$\boxed{\begin{aligned} x_1^3 - 2\sin x_3 &= \phi \\ x_1 \cdot x_2^2 - 1 &= \phi \\ &\vdots \\ x_1 \cdot x_{10} - x_k^2 &= \phi \end{aligned}}$$

- Initial guess. $\boxed{\underline{X}_0}$

- First approximation. $$\boxed{\underline{X}_1 = \underline{X}_0 - \underline{\underline{J}}^{-1}(\underline{X}_0) \cdot \underline{f}(X_0)}$$

$\underline{\underline{J}}$ is the Jacobian of the system.

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \vdots & & & \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{bmatrix}$$

- Second approximation. $$\boxed{\underline{X}_2 = \underline{X}_1 - \underline{\underline{J}}^{-1}(\underline{X}_1) \cdot \underline{f}(X_1)}$$

$$\vdots$$

- General.

$$\boxed{\underline{X}_{n+1} = \underline{X}_n - \underline{\underline{J}}^{-1}(\underline{X}_n) \cdot \underline{f}(X_n)}$$

Relabeling the unknown variables.

- Rearrange the equations $\boxed{\underline{f}(x) = \phi}$
- Relabel the variables. $P_1 \to x_1$

  $P_2 \to x_2$

  $\vdots$

- Form a vector from the unknown variables.

$$\underline{X} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ Q_{I} \\ Q_{II} \\ Q_{III} \\ Q_{IV} \\ Q_{V} \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

- Referring in a program code: $P_1 \to X(1) \equiv X_1$

$1\,Pa = \frac{1}{100.000}\,bar$

f_1: $[m^3/h]$ $\boxed{X(5) - X(6) - X(7) - d_2} = \phi$

f_2: $[m^3/h]$ $\boxed{X_7 - X_8 - d_3} = \phi$

f_3: $[m^3/h]$ $\boxed{X_8 + X_6 - X_9 - d_4} = \phi$

f_4: $[bar]$ $\boxed{X_1 - P_0} = \phi$

f_5: $[bar]$ $\boxed{\underbrace{(X_2 - X_1)}_{bar} - \rho \cdot g\, \underbrace{H_P}_{Pa}} \Big/ 100.000 = \phi$ $\qquad H_P(X_5)$

$$(X_2 - X_1) + \cancel{\frac{8 \cdot \rho}{\pi^2}\left(\frac{1}{D_2^4} - \frac{1}{D_1^4}\right) \cdot X_5^2} + \cancel{\rho \cdot g(h_2 - h_1)} - \rho \cdot g\, H_P = \phi$$

neglecting

Dimensions $\quad 1\,mm = \frac{1}{1000\,m}$

$P - [bar] \qquad 1\,\frac{m^3}{h} = \frac{1}{3600}\,\frac{m^3}{s}$

$Q - [m^3/h]$

$$\lambda_2 \frac{8 \overset{[m]}{L_2} \overset{[kg/m^3]}{\rho}}{\underset{[1/mm]^5}{D_2^5 \pi^2}} \underset{[m^3/h]}{X_6|X_6|} \cdot \frac{1.000.000}{1296}\ [bar] \quad \sim 771.6$$

$?$ — bar

f_6: $[bar]$ $$\boxed{(X_4 - X_2) + \lambda_2 \frac{8 L_2 \rho}{D_2^5 \pi^2} \cdot X_6|X_6| + \frac{\rho g (h_4 - h_2)}{100.000}} = \phi \qquad \frac{1000000}{1296}$$

f_7: $[bar]$ $$\boxed{(X_3 - X_2) + \lambda_3 \frac{8 L_3 \rho}{D_3^5 \pi^2} X_7|X_7| + \frac{\rho \cdot g (h_3 - h_2)}{100.000}} = \phi$$

f_8: $[bar]$ $$\boxed{(X_4 - X_3) + \lambda_4 \frac{8 L_4 \cdot \rho}{D_4^5 \pi^2} \cdot X_8|X_8| + \frac{\rho \cdot g (h_4 - h_3)}{100.000}} = \phi$$

$$\frac{m \cdot \frac{kg}{m^3}}{mm^5} \cdot \left[\frac{m^3}{h}\right]^2$$

f_9: $[bar]$ $$\boxed{(P_0 - X_4) + \frac{\rho \cdot g (h_b + h_w - h_4)}{100.000}} = \phi$$

- Unknowns: $\boxed{X_1, X_2, \dots, X_9}$
- Parameters: $\boxed{d_i}$ = Demands ; $\boxed{\lambda_i ; L_i ; D_i}$

  $\boxed{P_0, \rho, g}$ = Ambient pressure, density, gravity

  $\boxed{h_b, h_w}$ = Reservoir bottom height and water level.

  $\boxed{H_P(X_5)}$ = Pump characteristic curve.

- ~~Matlab / Code~~

- Daily schedule

HOMEWORK 1.

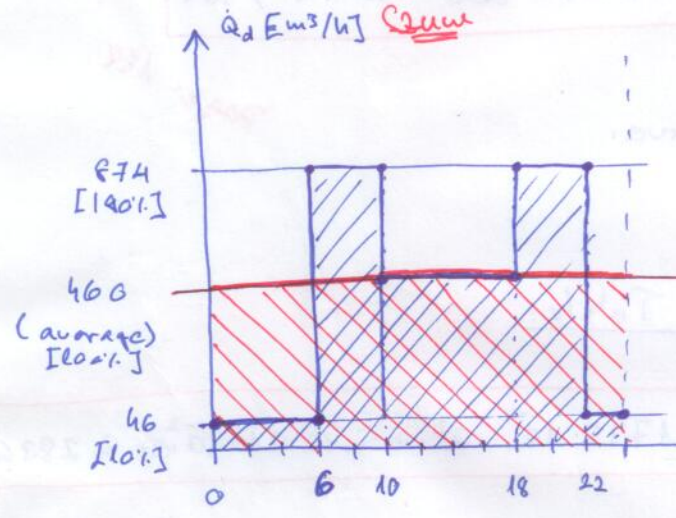
- How can we choose a pump?
- Reservoir sizing due to the daily fluctuations.

Total:

Average Consumption: $460\ m^3/h$

Pipe Network Solver.m

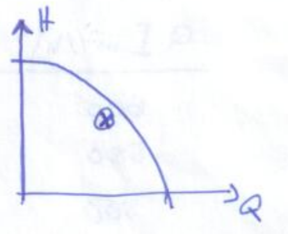
Daily consumption



- Is $460\ \frac{m^3}{h}$ the average.

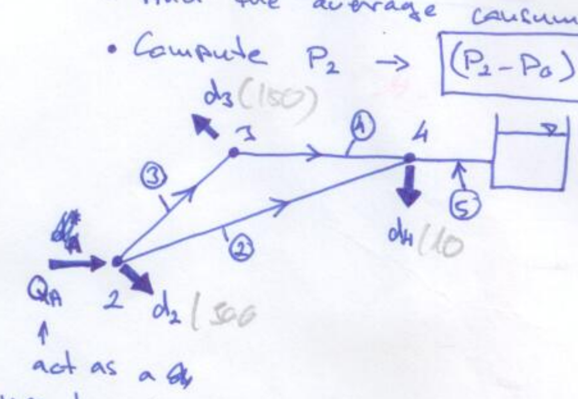
$$Q_A = \frac{6 \cdot 46 + 4 \cdot 874 + 8 \cdot 460 + 4 \cdot 874 + 2 \cdot 46}{24} =$$

$$\boxed{= 460\ m^3/h}$$



a) Need to choose a pump for the average consumption!

- Remove the pump
- Add the average consumption as an inlet of the system.
- Compute P_2 → $\boxed{(P_2 - P_0)/\rho \cdot g = H_s \text{ (required)}}$



- Equations Eq. Sys. 2. (NODES)

$$* \quad -Q_3 - Q_2 - \underbrace{d_2 + Q_A}_{\text{new}} = \phi$$

$$* \quad \boxed{d_2 = d_2 - Q_A}$$

$$Q_3 - Q_4 - d_3 = \phi$$

$$Q_4 + Q_2 - Q_5 - d_4 = \phi$$

- Relabel!!!
- Change the new demand property.
- Missing pump equation
- Missing B.C. at NODE 1.

$$\underline{X} = \begin{bmatrix} P_2 \\ P_3 \\ P_4 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} \quad \text{7 unknowns}$$

(Pipes)

$$P_2 - P_4 = \rho \cdot g(h_4 - h_2) + \lambda_2 \frac{8 L_2 \rho}{D_2^5 \pi^2} Q_2 |Q_2|$$

$$P_2 - P_3 = \ldots$$

$$P_3 - P_4 = \ldots$$

(reservoir)

$$\boxed{P_4 - P_0 = \rho g (h_0 + h_w - h_4)}$$

• The resulted pressure:

$$\boxed{P_2 = 7{,}0845 \text{ bar}}$$

$$P_0 = 1 \text{ bar}.$$

$$H_p = \frac{P_2 - P_0}{\rho \cdot g} = \frac{6{,}0845 \cdot 10^5}{1000 \cdot 9{,}81} \cong \underline{62 m}$$

$$Q_p = \underline{460 \text{ m}^3/\text{h}}$$

↓

$$\boxed{\text{Grundfos NBG/NKG 200-150-400/361}}$$

page: 188

• Extracting the characteristic curve:

Q [m³/h]	H [m]
600	52
550	57
500	60
450	63
400	65
350	67
300	68
250	69
200	69
150	68
100	67

Excel Table:

$$\boxed{H_p = -1{,}1795 \cdot 10^{-4} \cdot Q^2 + 5{,}3291 \cdot 10^{-2} \cdot Q + 6{,}2836 \cdot 10^1}$$

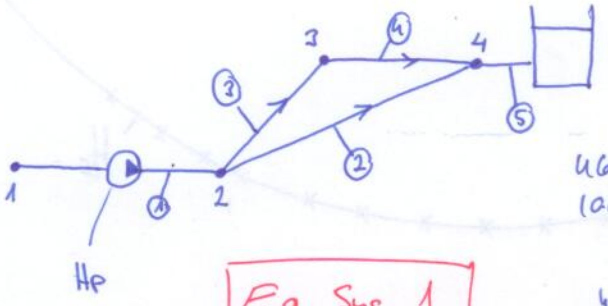
H [m]

Q [m³/h]

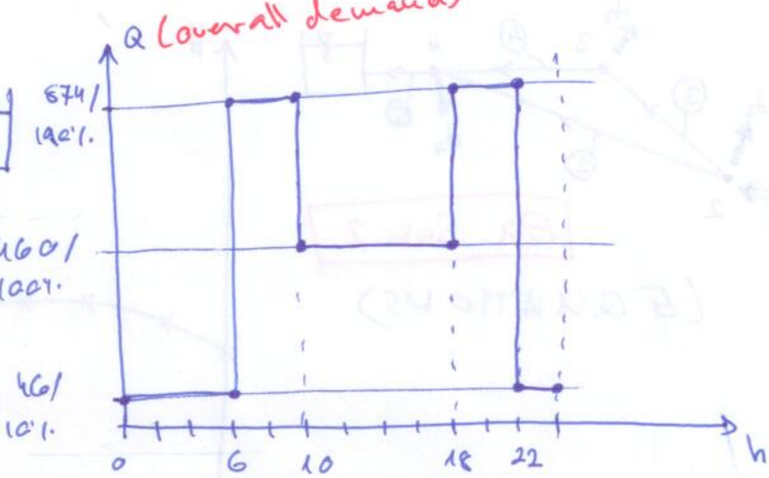
in Matlab: $-1{,}1795 \cdot 10^{-4}$ =

$-1{,}1795e\{-4\}$

b) Sizing the reservoir

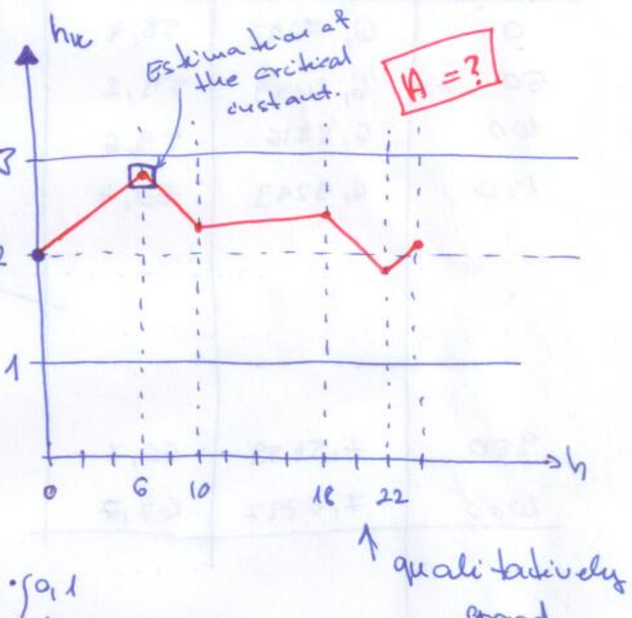


Eq. Sys. 1



• Determine the reservoir flow rate (EQUATIONS)

Time [h]	Q_5 [m³/h]	Water Level	Δh
0 – 6 (10%)	371,7	↑↑	
6 – 10 (190%)	– 373,6	↓↓	
10 – 18 (100%)	6,4	↑	
18 – 22 (190%)	– 373,6	↓↓	
22 – 24 (10%)	371,7	↑↑	



Parameters. Demand = [0 300 150 10] · $\begin{cases} 0,1 \\ 1 \\ 1,9 \end{cases}$

• $\begin{cases} Q_5 = 371,7\ [m^3/h] \\ \Delta t = 6h \\ \Delta h = 1m \end{cases}$ $Q_5 = \dfrac{A_R \cdot \Delta h}{\Delta t} \rightarrow A_R = \dfrac{Q_5 \cdot \Delta t}{\Delta h} \approx \underline{\underline{2230{,}2\,m^2}}$

$\boxed{A_R = 2300\,m^2}$

↓

$\Delta h_{0-6} = \dfrac{Q_5 \cdot \Delta t}{A_R} = \underline{\underline{0{,}97\,m}}$

$\Delta h_{6-10} = \dfrac{-373{,}6 \cdot 4}{2300} = -\underline{\underline{0{,}65\,m}}$

$\Delta h_{10-18} = \dfrac{6.4 \cdot 8}{2300} = 0{,}\underline{\underline{022}}\,m$

$\Delta h_{18-22} = \dfrac{-373{,}6 \cdot 4}{2300} = -\underline{\underline{0{,}65}}\,m$

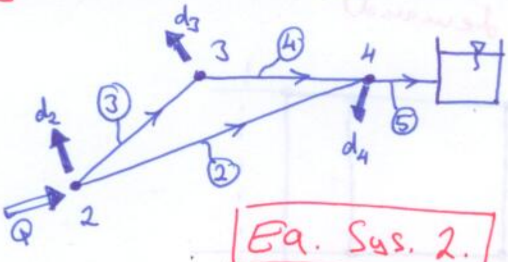
$\Delta h_{22-24} = \dfrac{371{,}7 \cdot 2}{2300} = 0{,}\underline{\underline{32}}\,m$

• Reservoir water levels:

h	hw [m]	
0	2	
6	2,97	
10	2,32	
18	2,34	
22	1,69	
24	2,01	

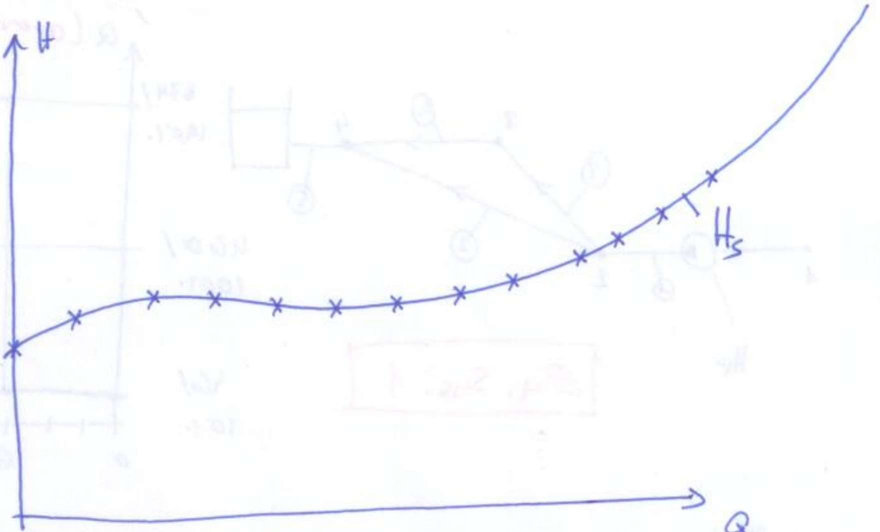
→ **Excel, Diagram !!!**

c) Characteristic curve of the System.



Eq. Sys. 2.

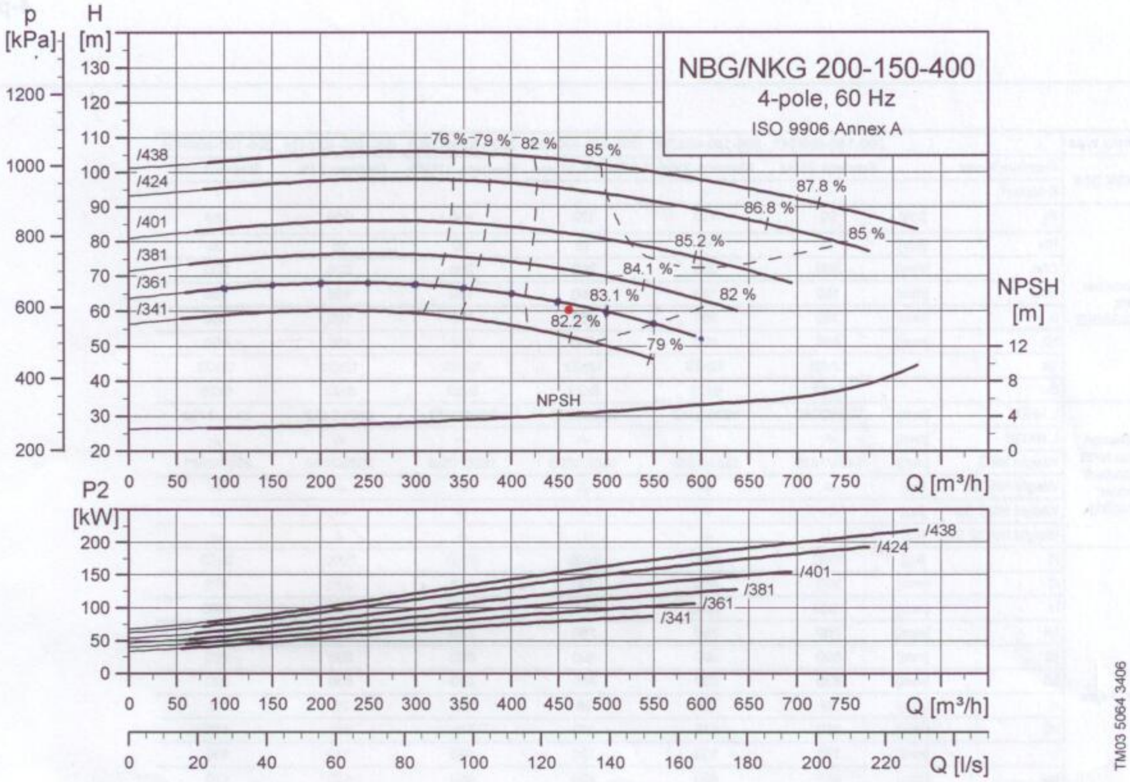
(EQUATIONS)

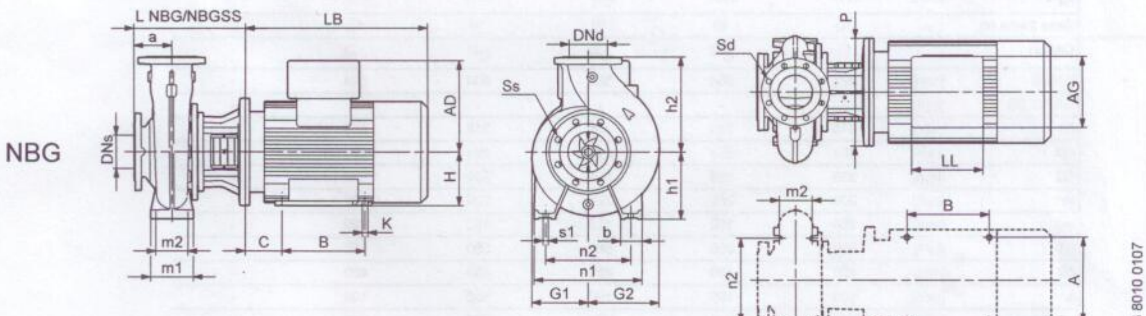


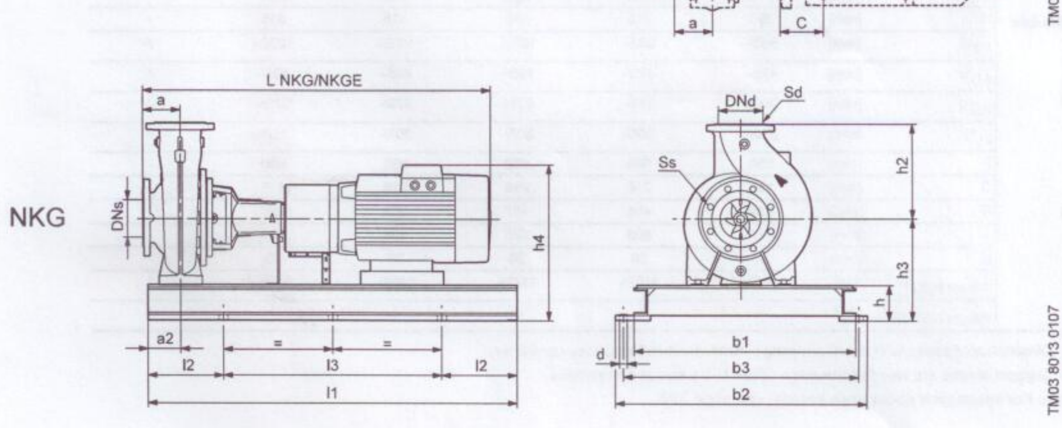
Q [m3/h]	P_2 [bar]	$\frac{P_2 - P_0}{\rho \cdot g}$ [m]
0	6,7293	58,4
50	6,8049	59,2
100	6,8716	59,6
150	6,9293	60,4
.		
.		
.		
.		
950	7,5459	66,7
1000	7,6397	67,7

Excel; Diagram.

Performance curves







Technical data

Pump type			200-150-400/341	200-150-400/361	200-150-400/381	200-150-400/401	200-150-400/424	200-150-400/438
Motor type	Premium Motor		Siemens 280M	Siemens 315S	Siemens 315MA	Siemens 315MB	Siemens 315L	Siemens 315
	E-Motor		-	-	-	-	-	-
Common data NBG/NKG	P_2	[kW]	90	110	132	160	200	288
	PN	[bar]	16	16	16	16	16	16
	DNs	[mm]	200	200	200	200	200	200
	DNd	[mm]	150	150	150	150	150	150
	a	[mm]	160	160	160	160	160	160
	h2	[mm]	450	450	450	450	450	450
	Ss		12x23	12x23	12x23	12x23	12x23	12x23
	Sd		8x23	8x23	8x23	8x23	8x23	8x23
Common data NKG standard/ spacer coupling	L NKG	[mm]	1904/2080	1936/2112	2096/2272	2096/2272	2236/2412	2244/2420
	L NKGE	[mm]	-/-	-/-	-/-	-/-	-/-	-/-
	Weight NKG	[mm]	1424/1419	1531/1535	1687/1690	1826/1830	2026/2030	2067/2071
	Weight NKGE	[kg]	-/-	-/-	-/-	-/-	-/-	-/-
	Weight NKG SS	[kg]	-/-	-/-	-/-	-/-	-/-	-/-
	Weight NKGE SS	[kg]	-/-	-/-	-/-	-/-	-/-	-/-
NKG data	l1	[kg]	2000	2000	2000	2000	2000	2250
	l2	[mm]	330	330	330	330	330	375
	l3	[mm]	1340	1340	1340	1340	1340	1500
	b1	[mm]	750	750	750	750	750	840
	b2	[mm]	890	890	890	890	890	980
	b3	[mm]	830	830	830	830	830	920
	d	[mm]	28	28	28	28	28	28
	a2	[mm]	110	110	110	110	110	110
	h	[mm]	130	130	130	130	130	130
	h3	[mm]	445	450	450	450	450	450
	h4[1]	[mm]	877/-	945/-	945/-	945/-	945/-	918/-
	Base frame no.		10	10	10	10	10	11
NBG data	Design		C	C[2]	C[2]	C[2]	C[2]	-
	L NBG	[mm]	474	504	504	504	504	-
	L NBG SS	[mm]	-	-	-	-	-	-
	h1	[mm]	315	315	315	315	315	-
	G1	[mm]	291	291	291	291	291	-
	G2	[mm]	339	339	339	339	339	-
	m1	[mm]	200	200	200	200	200	-
	m2	[mm]	150	150	150	150	150	-
	n1	[mm]	550	550	550	550	550	-
	n2	[mm]	450	450	450	450	450	-
	b	[mm]	100	100	100	100	100	-
	s1	[mm]	M20	M20	M20	M20	M20	-
	H	[mm]	280	315	315	315	315	-
	LB[1]	[mm]	930/-	932/-	1092/-	1092/-	1232/-	-/-
	AD[1]	[mm]	432/-	495/-	495/-	495/-	495/-	-/-
	AG[1]	[mm]	300/-	379/-	379/-	379/-	379/-	-/-
	LL[1]	[mm]	236/-	307/-	307/-	307/-	307/-	-/-
	P	[mm]	550	660	660	660	660	-
	C	[mm]	190	216	216	216	216	-
	B	[mm]	419	406	457	508	457	-
	A	[mm]	457	508	508	508	508	-
	K	[mm]	24	28	28	28	28	-
	Weight NBG[1]	[kg]	991/-	1167/-	1322/-	1462/-	1662/-	-/-
	Weight NBG SS[1]	[kg]	-/-	-/-	-/-	-/-	-/-	-/-

1) Dimension of pump with premium range motor/built-in frequency converter.
2) Support blocks are needed because of the P, h1 and H dimensions.
Note: For information about base frames, see page 222.

```
function PipeNetworkSolver
clc; clear all;

Parameters.Demand=[0 300 150 10]; %[m3/h]
Parameters.NodalHeight=[0 0 10 30 55]; %[m]
Parameters.FrictionCoefficients=[0 0.018 0.020 0.025 0]; %[-]
Parameters.PipeLength=[0 2000 1000 3400 0]; %[m]
Parameters.PipeDiameter=[0 400 300 200 0]; %[mm]
Parameters.AmbientPressure=1; %[bar]
Parameters.Density=1000; %[kg/m3]
Parameters.Gravity=9.81; %[m/s2]
Parameters.ReservoirBottomHeight=60; %[m]
Parameters.ReservoirWaterLevel=2; %[m]

InitialGuess=[3 3 3 3 250 250 250 250 250]'; %[bar] or [m3/h]


Results=SolveThePipeNetworkSystem(InitialGuess,Parameters);
fprintf('The data:\n');
    for k=1:length(Results)
        disp(Results(k));
    end



function f=SystemOfEquations(x,Parameters)
f=zeros(9,1);

f(1)=x(5)-x(6)-x(7)-Parameters.Demand(2);
f(2)=x(7)-x(8)-Parameters.Demand(3);
f(3)=x(8)+x(6)-x(9)-Parameters.Demand(4);

f(4)=x(1)-Parameters.AmbientPressure;

f(5)=(x(2)-x(1)) - Parameters.Density*Parameters.Gravity*PumpCharacteristicCurve(x(5))↙
/100000;

f(6)=(x(4)-x(2)) + Parameters.Density*Parameters.Gravity*(Parameters.NodalHeight(4)-↙
Parameters.NodalHeight(2))/100000 + ...
    8*Parameters.FrictionCoefficients(2)*Parameters.PipeLength(2)*Parameters.Density*x↙
(6)*abs(x(6))/Parameters.PipeDiameter(2)^5/pi^2 * 771.6;
f(7)=(x(3)-x(2)) + Parameters.Density*Parameters.Gravity*(Parameters.NodalHeight(3)-↙
Parameters.NodalHeight(2))/100000 + ...
    8*Parameters.FrictionCoefficients(3)*Parameters.PipeLength(3)*Parameters.Density*x↙
(7)*abs(x(7))/Parameters.PipeDiameter(3)^5/pi^2 * 771.6;
f(8)=(x(4)-x(3)) + Parameters.Density*Parameters.Gravity*(Parameters.NodalHeight(4)-↙
Parameters.NodalHeight(3))/100000 + ...
    8*Parameters.FrictionCoefficients(4)*Parameters.PipeLength(4)*Parameters.Density*x↙
(8)*abs(x(8))/Parameters.PipeDiameter(4)^5/pi^2 * 771.6;

f(9)=(Parameters.AmbientPressure-x(4)) + Parameters.Density*Parameters.Gravity*↙
(Parameters.ReservoirBottomHeight+Parameters.ReservoirWaterLevel-Parameters.NodalHeight↙
(4))/100000;
```

$$\frac{\cancel{m}\cdot\frac{kg}{m^{\cancel{3}2}}}{mm^5}\cdot\frac{m^6}{h^2}=$$

$$=\frac{\frac{kg}{m^2}}{mm^5}\cdot\frac{m^6}{h^2}=\frac{kg\cdot m^{\cancel{6}4}}{mm^5\cdot\cancel{m^2}\,h^2}=$$

$$=\frac{kg}{mm^5\,h^2}=\frac{1000^5}{1000^5}\,\frac{kg\,m^4}{mm^5\,h^2}=$$

$$(1000\,mm)^5$$

$$=1000^5\,\frac{kg\,m^4}{m^5\cdot h^2}=1000^5\,\frac{kg}{m\cdot h^2}=$$

$$=1000^5\,\frac{kg}{m\cdot(3600\,s)^2}=\frac{1000^5}{3600^2}\,\frac{kg}{m\cdot s^2}$$

$$Pa$$

$$=10^{15}$$

$$=\frac{1000^5}{3600^2}\cdot Pa=\frac{1000^5}{3600^2}\cdot\frac{1}{10^5}\,bar=$$

$$=\frac{10^{\cancel{15}\,10}}{\cancel{10^5}\,3600^2}\cdot bar=\frac{10^{10}}{3600^2}\,bar=$$

$$=\frac{10^{10}}{3.6^2\cdot10^6}\,bar=\frac{10^4}{3.6^2}\,bar=771.604$$

$$\boxed{=\frac{10.000}{12.96}}\cong 771.6$$

$$3600\cdot3600=3.6\cdot1000\cdot3.6\cdot1000=3.6^2\cdot10^6$$

```
function H = PumpCharacteristicCurve(Q)
% H[m], Q[m3/h]

H=250-0.0002*Q^2;


function x=SolveThePipeNetworkSystem(InitialGuess,Parameters)
ErrorTolerance=1e-6;
MaximumIteration=100;

[Error, Index]=max(abs(SystemOfEquations(InitialGuess,Parameters)));

NumberOfSteps=1; xOld=InitialGuess;
while Error>ErrorTolerance
    Jacobian=NumericalJacobian(xOld,Parameters);
    xNew=xOld-inv(Jacobian)*SystemOfEquations(xOld,Parameters);

    [Error, Index]=max(abs(SystemOfEquations(xNew,Parameters)));
        fprintf('%2d: Error=%5.3e, %2d\n',NumberOfSteps,Error,Index);

    xOld=xNew; x=xNew;
    NumberOfSteps=NumberOfSteps+1;

    if NumberOfSteps==MaximumIteration+1
        fprintf('Too many number of iteration, please try an other initial guess!!!
\n');
        Error=ErrorTolerance/2;
    end
    if Error<ErrorTolerance
        fprintf('Iteration converged!!!\n');
    end
end

function dfdx=NumericalJacobian(x,Parameters)
dfdx=zeros(length(x));
f=SystemOfEquations(x,Parameters);
    for i=1:length(x)
        x1=x;
        dx=x1(i)*0.001;
        if abs(dx)<1e-6
            dx=1e-6;
        end
        x1(i)=x1(i)+dx;
        f1=SystemOfEquations(x1,Parameters);
        dfdx(:,i)=(f1-f)/dx;
    end
```

• Compressible Flows

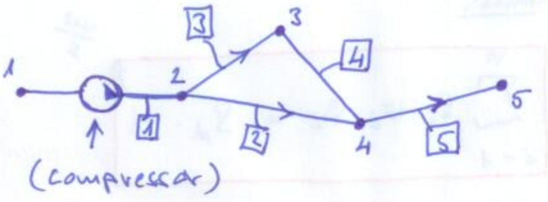
(4)

- a more general treatment of pipe network systems.

• Basic principles: • Mass
• Momentum
• Energy

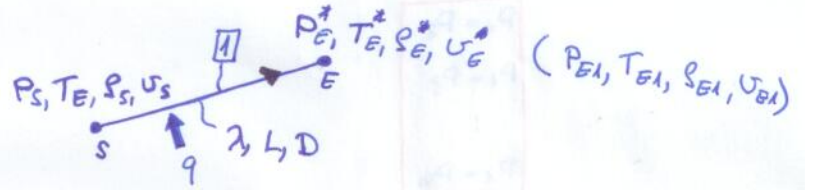
• Model:

For Instance.



• Equation system

• BRANCH EQUATIONS



PIPES

• Mass:

$$\boxed{A_S \cdot v_S \rho_S = A_E v_E \rho_E}$$

• Momentum:

$$\boxed{P_S - P_E = \frac{1}{2}\left(\rho_E \cdot v_E^2 - \rho_S \cdot v_S^2\right) + g\left(\rho_E \cdot h_E - \rho_S h_S\right)}\ \boxed{+\Delta p'}$$

• Energy:

$$\boxed{c_p\left(T_E - T_S\right) + \frac{1}{2}\left(v_E^2 - v_S^2\right) + g\left(h_E - h_S\right) = q}$$

• Equation of state:

for instance the ideal gas law

$$\boxed{\frac{P_S}{\rho_S} = R \cdot T_S} \qquad \boxed{\frac{P_E}{\rho_E} = R \cdot T_E}$$

• Compressor:

$$A_S v_S \rho_S = A_E v_E \rho_E$$

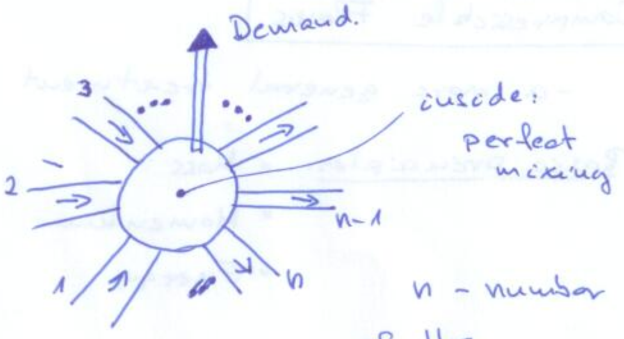
$$P_E - P_S = f(Q) \quad \text{- characteristic curve.}$$

$$\frac{P_E}{\rho_E^{\kappa}} = \frac{P_S}{\rho_S^{\kappa}} \quad \text{- adiabatic compression.}$$

κ - adiabatic exponent $\left(\kappa = \frac{c_p}{c_v}\right)$

for diatomic gases $\kappa \cong 1.4$

• NODAL EQUATIONS



n - number of the connecting pipes

• Mass

$$\boxed{\sum_{i=1}^{n} \rho_i\, v_i\, A_i = \rho_d \cdot d} \quad \frac{m^3}{s}$$

• Momentum (no pressure jump)

$$\boxed{\begin{aligned} p_1 &= p_2 \\ p_1 &= p_3 \\ &\vdots \\ p_1 &= p_n \end{aligned}}$$

• Energy (~~no temperature jump~~)

$$\boxed{\cancel{\begin{aligned} T_1 &= T_2 \\ &\vdots \\ T_1 &= T_n \end{aligned}}}$$

perfect mixing.

$$\boxed{\sum_{i=1}^{k} c_p \cdot \dot m_i\, T_i = c_p \cdot \dot M\, T^*}$$

T^* - ~~average~~ mixture temperature

k = number of inflow (we don't know in advance)

$\dot M$ - total in mass flow.

$$\boxed{T^* = T_{k+1};\ T^* = T_{k+2};\ T^* = T_n}$$

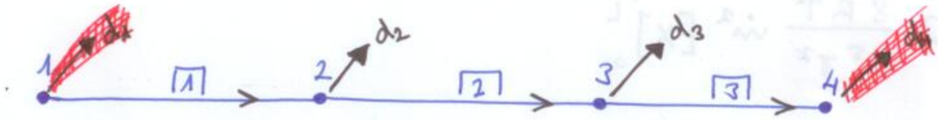
• Boundari conditions

$$\boxed{\begin{aligned} &T_1 = \dots \\ &v_1 = \dots \quad \text{or} \quad p_1 = \dots \end{aligned}}$$

every node connecting to a single pipe need 2 BC.-s

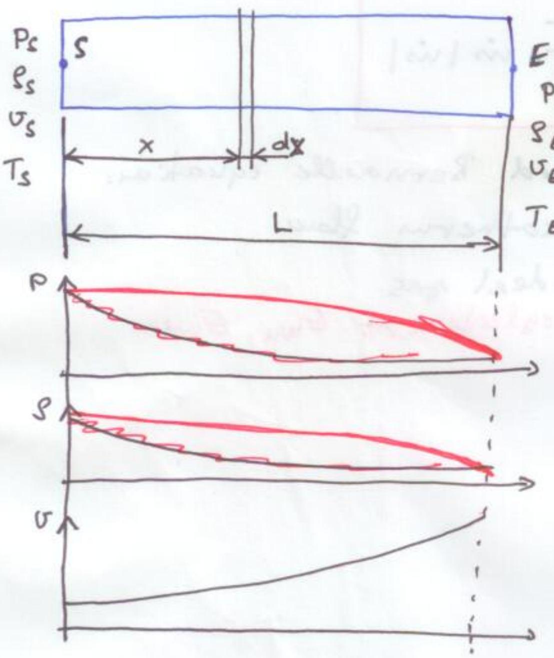
Natural Gas Pipelines

- Simplifications:
  - Isothermal case
  - We do not model the compressor.



a) Friction modeling for isothermal case:

(Modified Bernoulli equation)



The usual pressure loss

$$\boxed{\Delta p' = \lambda \cdot \frac{L}{D} \cdot \frac{\rho}{2} v^2}$$

- Problem: $\rho, v = f(x) \to Re(x) \to \lambda(x)$

- Bernoulli equation:

$$p_S - p_E = \underbrace{\frac{1}{2}\left(\rho_E v_E^2 - \rho_S v_S^2\right)} + \underbrace{g\left(\rho_E \cdot h_E - \rho_S h_S\right)} + \Delta p'$$

let us neglect these terms

$$\boxed{p_S - p_E = \Delta p'}$$ the pressure difference is due to the friction.

- Differential form for a small segment.

$$\boxed{-dp = \lambda \frac{dx}{D} \frac{\rho}{2} v^2}$$ $\rho(x)$, $v(x)$, $\lambda(x)$

- $\dot{m} = \rho \cdot A \cdot v \Rightarrow v = \frac{\dot{m}}{\rho \cdot A}$; $A = \frac{D^2 \pi}{4}$; $\rho = \frac{p}{RT}$ ideal gas law

$$-dp = \lambda \cdot \frac{dx}{D} \cdot \frac{\cancel{\rho}}{2} \frac{\dot{m}^2}{\rho^{\cancel{2}} A^2}$$

$$-dp = \lambda \cdot \frac{dx}{D} \frac{\dot{m}^2 \, \cancel{16}\, 8}{\cancel{2} \rho D^4 \pi^2}$$

$$-dp = \lambda \cdot \frac{dx}{D^5} \frac{8 \dot{m}^2 RT}{\pi^2 p}$$

$$\underbrace{-p \cdot dp}_{f(p)} = \boxed{\underbrace{\lambda \cdot \frac{8 RT \cdot \dot{m}^2}{D^5 \pi^2}}} dx \quad / \int$$

- assume isotherm flow

$T = \text{konst} \Rightarrow T_S = T_E = T$

$Re = \frac{v \cdot D}{\nu} = \frac{\dot{m} D}{\rho \cdot A \nu} = \frac{\dot{m} D}{A \cdot \mu} = \text{konst}$

$\mu = f(T)$ $\quad \lambda = \text{konst}$

$$-\int_{P_S}^{P_E} P\,dP = \lambda \cdot \frac{8RT}{D^5\pi^2} \cdot \dot{m}^2 \int_0^L 1\,dx$$

$$-\left[\frac{P^2}{2}\right]_{P_S}^{P_E} = \lambda \frac{8RT}{D^5\pi^2} \dot{m}^2 \left[x\right]_0^L$$

$$\frac{P_S^2 - P_E^2}{2} = \lambda \frac{8RT}{D^5\pi^2} \dot{m}^2 \cdot L$$

$$\boxed{\frac{P_S^2 - P_E^2}{2} = \lambda \frac{8RLT}{D^5\pi^2} \dot{m}|\dot{m}|}$$

modified Bernoulli equation:
- Isotherm flow
- Ideal gas
- neglection of G_{kin}, E_{pot}

b) Unknowns – Collecting the equations

P_1	ρ_1	v_{1S}
P_2	ρ_2	v_{1E}
P_3	ρ_3	v_{2S}
P_4	ρ_4	v_{2E}
		v_{3S}
		v_{3E}

$\dot{m}_1$
$\dot{m}_2$
$\dot{m}_3$

7 unknowns

- NODAL EQUATIONS:

$$\dot{m}_1 - \dot{m}_2 - d_2 = \phi$$
$$\dot{m}_2 - \dot{m}_3 - d_3 = \phi$$

- BRANCH EQUATIONS:

$$\frac{P_1^2 - P_2^2}{2} = \lambda_1 \frac{8RL_1T}{D_1^5\pi^2} \dot{m}_1|\dot{m}_1|$$

$$\frac{P_3^2 - P_2^2}{2} = \lambda_2 \frac{8RL_2T}{D_2^5\pi^2} \dot{m}_2|\dot{m}_2|$$

$$\frac{P_4^2 - P_3^2}{2} = \lambda_3 \frac{8RL_3T}{D_3^5\pi^2} \dot{m}_3|\dot{m}_3|$$

- BOUNDARY CONDITIONS

e.g.

$P_1 = \ldots$, $P_4 = \ldots$ or $\dot{m}_1 = \ldots$, $P_1 = \ldots$ (Problem 3) or $\dot{m}_1 = \ldots$, $P_4 = \ldots$ or $\dot{m}_3 = \ldots$, $P_1 = \ldots$ or $\dot{m}_3 = \ldots$, $P_4 = \ldots$

but ~~$\dot{m}_1 = \ldots$, $\dot{m}_3 = \ldots$~~

- OTHER ISSUES

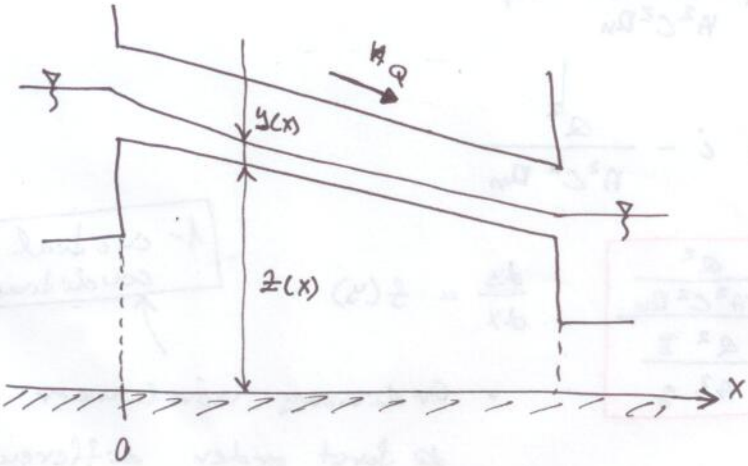
R – for a given material – (internet)

T – choose a temperature or $\boxed{T_x = \frac{P_x}{R \cdot \rho_x}}$ ideal gas

if we know the density at a point.

- Open channel flow

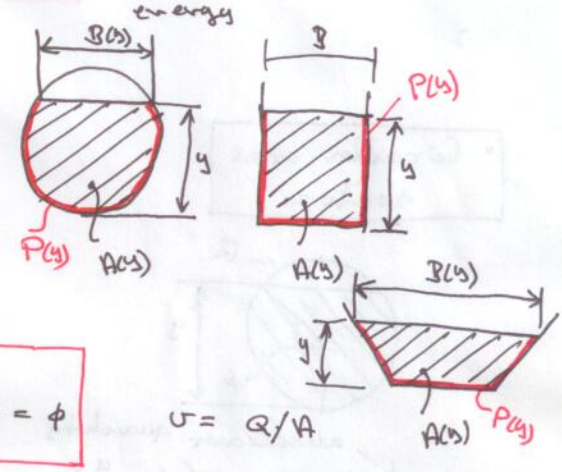
(5)



Total mechanical energy is constant:

$$\boxed{(z+y) + \frac{v^2}{2\cdot g} + h' = const}$$

potential energy — kinetic energy — friction loss



- The change of rate along the channel:

$$\frac{d}{dx}\left(z + y + \frac{v^2}{2g} + h'\right) = \phi$$

$$\boxed{\frac{dy}{dx} + \underset{①}{\frac{dz}{dx}} + \underset{②}{\frac{d}{dx}\left(\frac{v^2}{2\cdot g}\right)} + \underset{③}{\frac{dh'}{dx}} = \phi}$$

$v = Q/A$

$Q = v \cdot A$

- $\frac{dz}{dx} = \boxed{-i}$ (Sloap)

- $$\frac{d}{dx}\left(\frac{v^2}{2g}\right) = \frac{1}{2g}\frac{d}{dx}\left(\frac{Q^2}{A^2}\right) = \frac{Q^2}{2\cdot g}\frac{d}{dx}\left(\frac{1}{A^2}\right) = \frac{Q^2}{\cancel{2}g}\left(-\cancel{2}\frac{1}{A^3}\cdot\frac{dA}{dx}\right) =$$

A^{-2} ; $A = A(y)$

$$= -\frac{Q^2}{g\cdot A^3}\cdot\frac{dA}{dy}\frac{dy}{dx} = \boxed{-\frac{Q^2}{g\,A^3}\cdot B\cdot\frac{dy}{dx}}\ ;$$

$Q = const$
$A = f(y)$
$B = f(y)$

- $$\frac{dh'}{dx} = \frac{v^2}{C^2 R_h} = \boxed{\frac{Q^2}{A^2 R_h C^2}} =$$

R_h = hydraulic radius $\boxed{R_h = \frac{A}{P}}$; P - is the wetted perimeter.

C is the Chezy coefficient. $\boxed{C = R_h^{\frac{1}{6}} / n}$; n - is the Manning Coefficient of Roughness from tables

$\frac{R_h^{1/6}}{n}$ ozlen! GP

$$\frac{dy}{dx} - i - \frac{Q^2 B}{A^3 \cdot g}\frac{dy}{dx} + \frac{Q^2}{A^2 C^2 R_h} = \phi$$

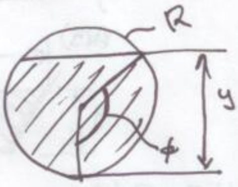
$$\frac{dy}{dx}\left(1 - \frac{Q^2 B}{A^3 \cdot g}\right) = i - \frac{Q^2}{A^2 C^2 R_h}$$

$$\boxed{\frac{dy}{dx} = \frac{i - \frac{Q^2}{A^2 C^2 R_h}}{1 - \frac{Q^2 B}{A^3 \cdot g}}} \qquad \frac{dy}{dx} = f(y)$$

1- critical condition.

- Ordinary nonlinear first order differential equation.

• Circular cross section



auxiliary quantity

$$\phi = \arccos\left(1 - \frac{y}{R}\right)$$

$$A = R^2(\phi - \sin\phi \cdot \cos\phi)$$

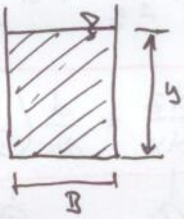
$$P = 2 \cdot R \cdot \phi = D \cdot \phi$$

$$B = 2 \cdot R \cdot \sin\phi$$

$$R_h = \frac{A}{P} = \frac{D}{4}\left(1 - \frac{\sin 2\phi}{2\phi}\right)$$

again: $\boxed{\begin{matrix} A(y) \\ B(y) \\ R_h(y) \end{matrix}}$

• Rectangular cross section.



$$B = \text{constant}$$

$$A = B \cdot y$$

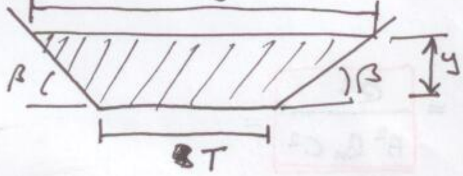
$$P = B + 2 \cdot y$$

$$R_h = \frac{A}{P} = \frac{B \cdot y}{B + 2 \cdot y}$$

$$\boxed{A(y);\ B(y);\ R_h(y)}$$

$B = B$; $B \neq B(y)$

• Trapezoidal cross section



$$\tan\beta = \frac{y}{(B-T)/2}$$

$$(B-T)\tan\beta = 2 \cdot y$$

$$B - T = \frac{2 \cdot y}{\tan\beta}$$

$$\boxed{A = y \cdot (B+T)/2}$$

$$\boxed{B = T + \frac{2 \cdot y}{\tan\beta}}$$

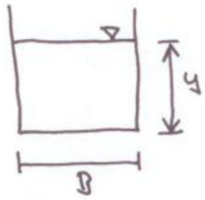
$$\boxed{P = T + 2 \cdot \sqrt{y^2 + \left(\frac{B-T}{2}\right)^2}}$$

$$\boxed{R_h = \frac{A}{P} = \dots}$$

- Specific feature: sound speed

ODE: $\dfrac{dy}{dx} = \dfrac{i - \dfrac{Q^2}{A^2 C^2 R_H}}{1 - \boxed{\dfrac{Q^2 \cdot B}{A^3 \cdot g}}}$

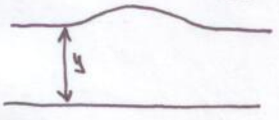
Specifically:



$\rightarrow \dfrac{A}{B} = y$

$$\frac{Q^2 \cdot B}{A^3 \cdot g} = \frac{v^2 \cdot B}{A \cdot g} = \boxed{\frac{v^2}{g \cdot y}}$$

- Without the proof, it can be derived that:



the celerity of a shallow water wave: (sound speed)

$$\boxed{a_{sw} = \sqrt{g \cdot y}}$$

- $\dfrac{Q^2 \cdot B}{A^3 \cdot g} = \dfrac{v^2}{a_{sw}^2} = \underline{\underline{Fr^2}}$ Froude number $Fr = \dfrac{v}{a_{sw}}$

- If $Fr < 1 \rightarrow$ subcritical flow $\underline{\underline{v < a_{sw}}}$

  If $Fr > 1 \rightarrow$ supercritical flow $\underline{\underline{v > a_{sw}}}$ **Mach number**

- When Fr increased from lower than 1 to greater 1 $\rightarrow$ special phenomenon (hydraulic jump) **Sonic boom**

VIDEO

- Numerical values:

  - 1mm depth: (wave speed)

    $a_{sw} = \sqrt{9.81 \cdot 1} \cong \underline{\underline{3,13}}$ m/s

  - Water: (filled pipe)

    $a_w \cong \underline{\underline{1400}}$ m/s

  - Air: (sound speed)

    $a_{air} = \sqrt{\kappa RT} \cong \underline{\underline{340}}$ m/s (airplane)

    (ideal gas)

  - Steel: (railroads)

    $a_{st} \cong \underline{\underline{6100}}$ m/s

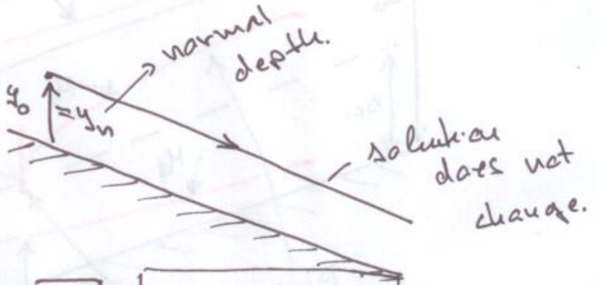
- Specific depths

$$\boxed{\frac{dy}{dx} = \frac{i - \frac{Q^2}{A^2 C^2 R_h}}{1 - \frac{Q^2 \cdot B}{A^3 \cdot g}}}$$

- Normal depth.

if $\underbrace{\boxed{i - \frac{Q^2}{A^2 C^2 R_h} = \phi}}_{f(y)} \Rightarrow \frac{dy}{dx} = \phi$

$f(y) \rightarrow \boxed{y_n = \dots}$



$\boxed{y_n}$ $\boxed{y_n = f(Q, i)}$

- Non linear function $\boxed{f(y) = \phi}$ for instance Newton-Rapson. 1Dimension.

- for rectangular channel:

$$i - \frac{Q^2}{B^2 \cdot y^2 \cdot R_h^{\frac{2}{6}} \cdot n^2 \cdot R_h} = \phi \qquad R_h = \frac{B \cdot y}{B + 2 \cdot y} \qquad \frac{1}{3} + 1 = \frac{1}{3} + \frac{3}{3} = \frac{4}{3}$$

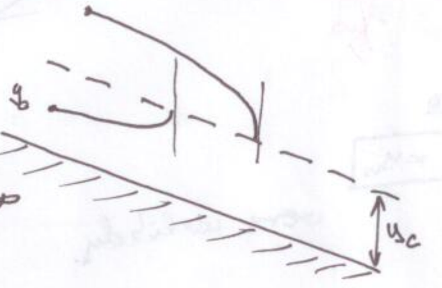
$$C = R_h^{\frac{1}{6}} \cdot n$$

$$i - \frac{Q^2}{B^2 \cdot y^2 \cdot n^2 \cdot R_h^{\frac{4}{3}}} = \phi$$

$$i - \frac{Q^2}{n^2 \cdot B^2 \left(\frac{B \cdot y}{B + 2 \cdot y}\right)^{\frac{4}{3}} \cdot y^2} = \phi \rightarrow \text{no analitc solution.}$$

- Critical depth:

if $\underbrace{\boxed{1 - \frac{Q^2 B}{A^3 \cdot g} = \phi}}_{f(y)} \Rightarrow \frac{dy}{dx} = \infty$



non linear equation

$\boxed{f(y) = \phi}$

Newton-Rapson.

$\boxed{y_c}$ - critical depth.

Problem: • Physically invalid solution
• Or phenomenon which cannot described with 1D equation (hydraulic jump)

- specifically for rectangular cross section.

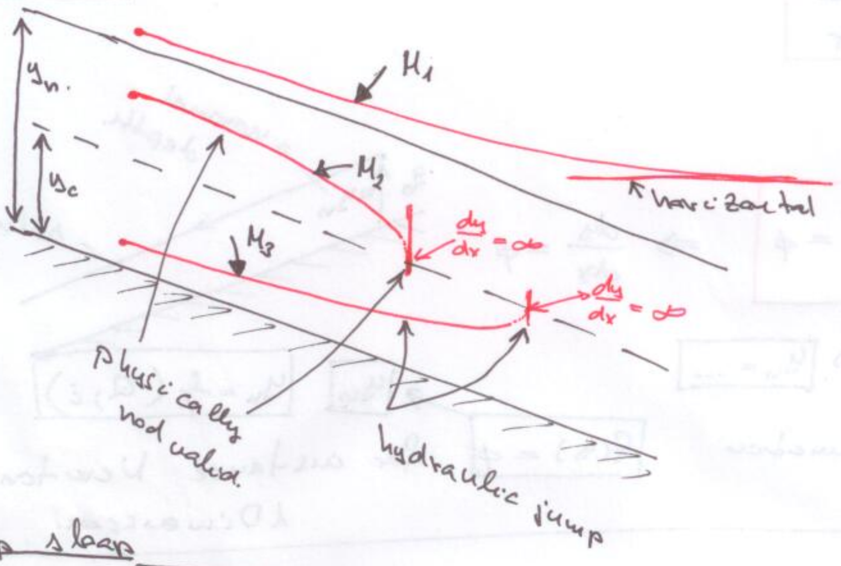
$$1 - \frac{Q^2 \cdot \cancel{B}}{B^{2}\cancel{B} \cdot y_c^3 \cdot g} = \phi \qquad y_c^3 = \frac{Q^2}{B^2 \cdot g} \qquad \boxed{y_c = \sqrt[3]{\frac{Q^2}{B^2 g}}}$$

$$1 = \frac{Q^2}{B^2 \cdot y_c^3 \cdot g} \qquad \boxed{y_c = f(Q)}$$

• Possible configurations

• Mild slope

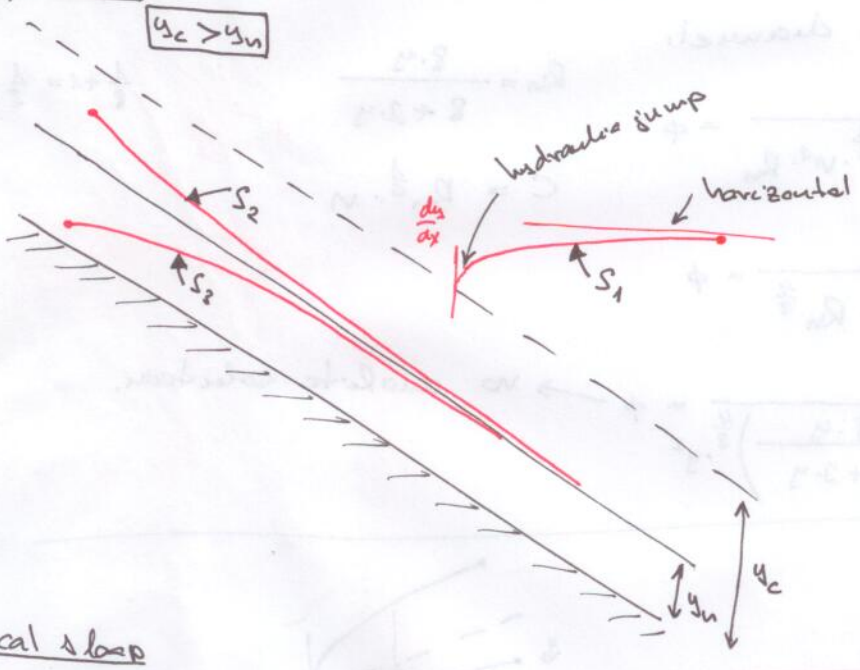
$\boxed{y_n > y_c}$



M_i - solutions of ODE

• Steep slope

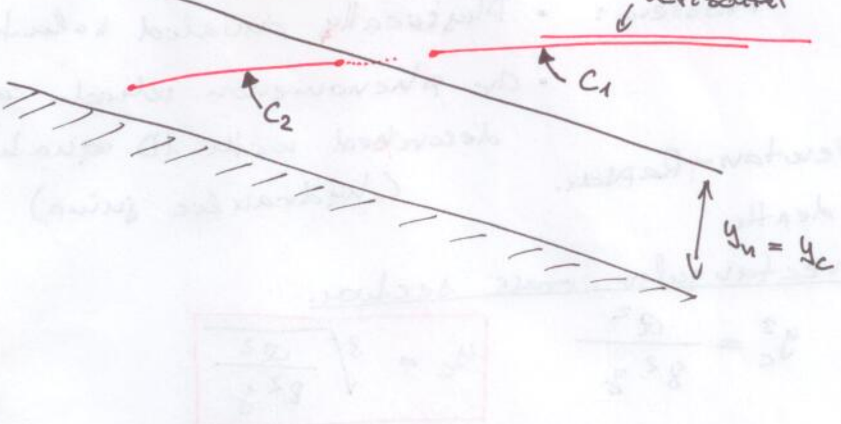
$\boxed{y_c > y_n}$



• Critical slope

$\boxed{y_c = y_n}$

very unlikely.



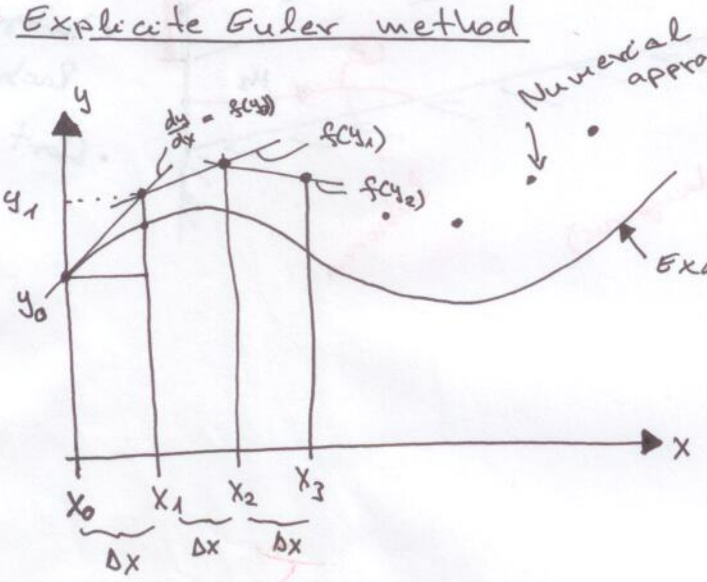
• Numerical solution technique

$$\frac{dy}{dx} = \frac{i - \frac{Q^2}{A^2 \cdot C^2 R_h}}{1 - \frac{Q^2 \cdot B}{A^3 \cdot g}} \quad ; \quad \boxed{\frac{dy}{dx} = f(y)}$$

• Nonlinear → only numerical solutions (approximation)

• First order → 1 initial condition y_0 needed.

• <u>Explicite Euler method</u>



$\boxed{\text{Forward in } X}$

$y_1 = y_0 + f(y_0) \cdot \Delta X \quad | \quad X_1 = X_0 + \Delta X$

$y_2 = y_1 + f(y_1) \cdot \Delta X \quad | \quad X_2 = X_1 + \Delta X$

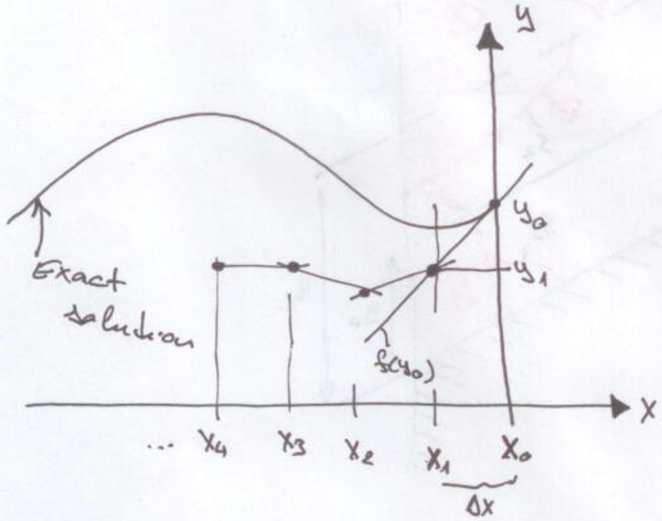
$y_3 = y_2 + f(y_2) \cdot \Delta X \quad | \quad X_3 = X_2 + \Delta X$

$y_4 = \ldots \quad | \quad X_4 = \ldots$

$\vdots \qquad \vdots$

Numerical solution $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$; $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$\boxed{\text{Backward in } X}$



$y_1 = y_0 - f(y_0) \cdot \Delta X \; ; \; X_1 = X_0 - \Delta X$

$y_2 = y_1 - f(y_1) \cdot \Delta X \; ; \; X_2 = X_1 - \Delta X$

$\vdots$

• If $\Delta X \to 0$ the approximation tends to the exact solution.

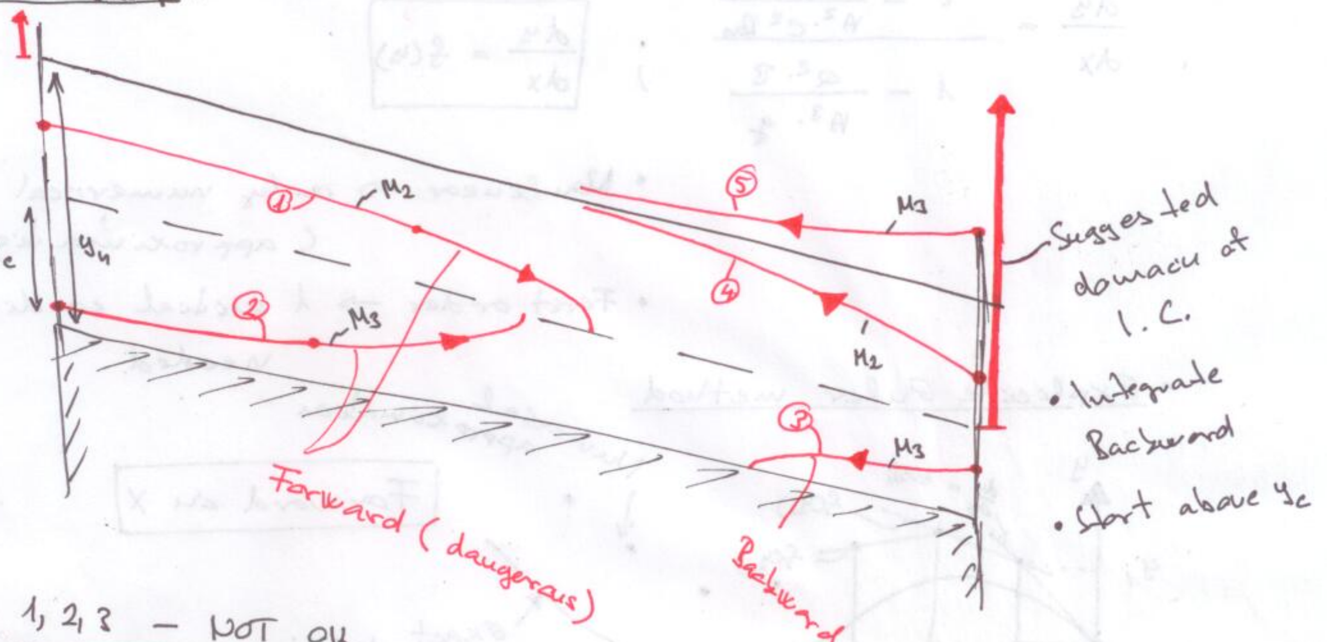
↓

• Try several number of ΔXs

• Decrease ΔX if necessary

• Suggestions.

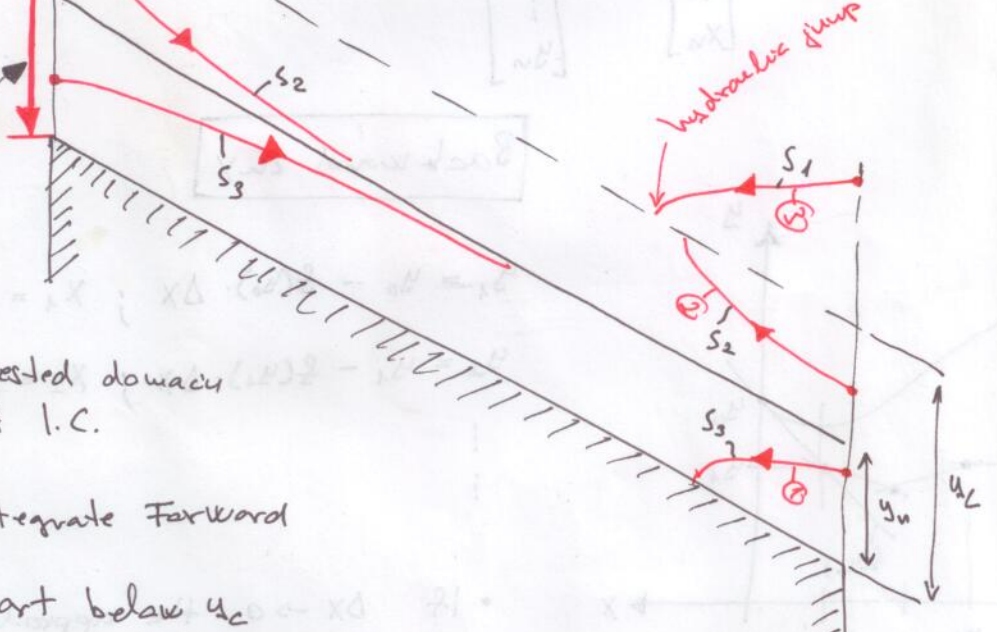
• Mild slope.



1, 2, 3 — NOT OK

$\boxed{4, 5 \text{ — OK}}$

• Steep slope



• Suggested domain of I.C.

• Integrate Forward

• Start below y_c