Introduction to mechanical engineering lecture notes

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Contents

1 A short summary of the basics

1.1 Physical quantities and their units

The value of a **physical quantity** Q is expressed as the product of a numerical value Q and a unit of measurement $[Q]$:

$$Q = Q \times [Q] \tag{1}$$

For example, if the temperature T of a body is quantified (measured) as 25 degrees Celsius this is written as:

$$T = 25 \times^{o} C = 25^{o}C, \tag{2}$$

where T is the symbol of the physical quantity "temperature", 25 is the numerical factor and ^{o}C is the unit.

By convention, physical quantities are organized in a dimensional system built upon base quantities, each of which is regarded as having its own dimension. The seven base quantities of the International System of Quantities (ISQ) and their corresponding SI units are listed in Table 1. Other conventions may have a different number of fundamental units (e.g. the CGS and MKS systems of units).

Name	Symbol for quantity	SI base unit	Abbreviation for unit
Length	l, x, r, etc.	meter	m
Time	t	second	s
Mass	m	kilogram	kg
Electric current	I, i	ampere	A
Thermodynamic temperature	T	kelvin	K
Amount of substance	n	mole	mol
Luminous intensity	I_v	candela	cd

Table 1: International System of Units base quantities

All other quantities are derived quantities since their dimensions are derived from those of base quantities by multiplication and division. For example, the physical quantity velocity is derived from base quantities length and time and has base unit m/s. Some derived physical quantities have dimension 1 and are said to be *dimensionless quantities.*

The units of some derived quantities have their own base units named by prominents of the history of physics. For example the base unit of the force is 1 Newton (abbreviated as N). 1N can be still expressed by the base units: $F = m \cdot a \rightarrow N = kg \cdot m/s^2$ (Where a is the acceleration with base unit m/s^2, see in later chapters).

The International System of Units (SI) specifies a set of unit *prefix*es known as SI prefixes or metric prefixes. An SI prefix is a name that precedes a basic unit of measure to indicate a decimal multiple or fraction of the unit. Each prefix has a unique symbol that is prepended to the unit symbol, see Table 2.

Prefix	Symbol	10^n
giga	G	10^9
mega	M	10^6
kilo	k	10^3
hecto	h	10^2
deca	da	10^1
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}

Table 2: International System of Units prefixes.

Examples: $mm = 10^{-3}m$, $MN = 10^6N$, $kg = 10^3g$. Note that in all cases the base unit is without any prefix expect for the mass, where the base unit is the kilogram, see Table 1.

Units can be used as numbers in the sense that you can add, subtract, multiply and divide them - with care. Much confusion can be avoided if you work with units as though they were symbols in algebra. For example:

- Multiply units along with numbers:

  (5 m) × (2 sec) = (5 × 2) × (m × sec) = 10 m sec.

  The units in this example are meters times seconds, pronounced as 'meter seconds' and written as 'm sec'.

- Divide units along with numbers:

  (10 m) / (5 sec) = (10 / 5) × (m / sec) = 2 m/sec.

  The units in this example are meters divided by seconds, pronounced as 'meters per second' and written as 'm/sec'. This is a unit of speed.

- Cancel when you have the same units on top and bottom:

  (15 m) / (5 m) = (15 / 5) × (m / m) = 3.

  In this example the units (meters) have cancelled out, and the result has no units of any kind! This is what we call a 'pure' number. It would be the same regardless what system of units were used.

- When adding or subtracting, convert both numbers to the same units before doing the arithmetic:

  (5 m) + (2 cm) = (5 m) + (0.02 m) = (5 + 0.02) m = 5.02 m.

  Recall that a 'cm', or centimeter, is one hundredth of a meter. So 2 cm = (2 / 100) m = 0.02 m.

- You can't add or subtract two numbers unless you can convert them both to the same units:

  (5 m) + (2 sec) = ???

1.2 Extensive and Intensive quantities

Physical quantities can be grouped to extensive ones and intensive ones.

Extensive: when its magnitude is additive for subsystems (volume, mass, etc). If You pour together 5 dl beer and another 5 dl beer You got $5dl + 5dl = 10dl$. Congratulations, it is already a Maßkrug.

Intensive: when the magnitude is independent of the extent of the system (temperature, pressure, etc.) Assume that the first jug of the beer has a temperature of $5°C$ and the other one has also $5°C$. Fortunately the result is not a $10°C$ beer if You pour them together because it would be already too hot to drink it.

1.3 Scalar and Vector quantities

Another possibility of grouping physical quantities are as follows:

Scalar: when a physical quantity has only a magnitude but no direction (pressure, temperature, volume, mass). Note that scalar quantities can also have sign ($-5°C or + 10°C$), but no exact direction in the (2 or 3 dimensional) space.

Vector: when a physical quantity has both magnitude and direction (force, velocity). For example if You fly from Budapest with 800km/h, it does matter in which direction You fly (i.e. your velocity vectors have also direction): You will reach different locations when having velocities of different directions. Vector quantities are usually typeset with bold or underscored: $\mathbf{F}$, $\mathbf{v}$ or $\underline{F}$, $\underline{v}$

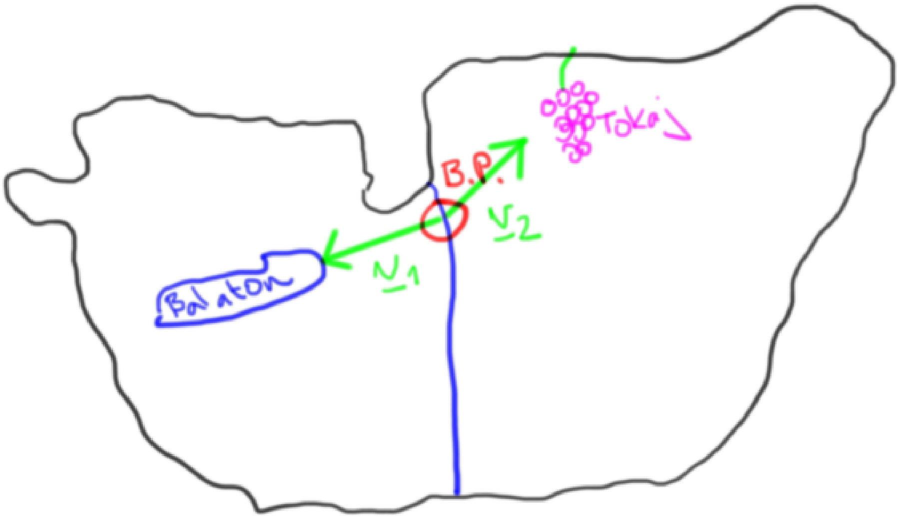


Figure 1: Velocity is a vector quantity: it has a direction not only magnitude

With vectors simple operations can be done. They can be added by connecting the tip and tail of two vectors and drawing in the resulting vector.

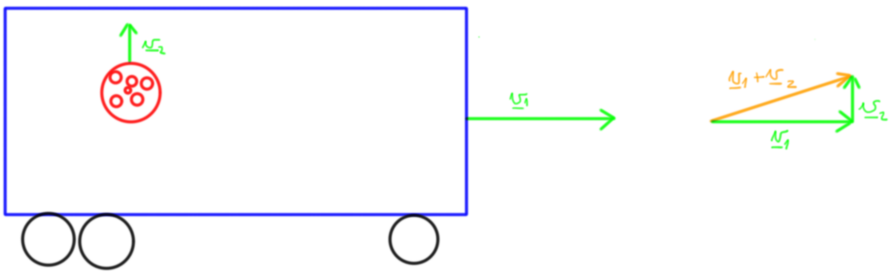


Figure 2: Adding velocity vectors. $\mathbf{v_1}$ is the velocity of the bus, $\mathbf{v_2}$ is the velocity of the ball relative to the bus. The resulting vector (orange) is the velocity of the ball by an observer standing outside of the bus.

Subtracting works very similar to addition with the only difference that the vector with the negative sign shall be first reversed.

Figure 3: Subtracting vectors. $\mathbf{v_2}$ is reversed first then a simply addition is performed.

The dot product or scalar product of two vectors are defined as

$$\mathbf{v_1} \cdot \mathbf{v_2} = |\mathbf{v_1}|\,|\mathbf{v_2}| \cos\theta = v_1 v_2 \cos\theta, \tag{3}$$

where θ is the angle of the two vectors and $||$ denotes the absolute value of the vector which is practically the length of the vector.

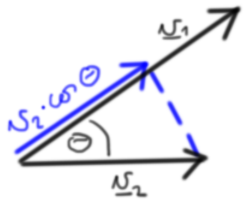


Figure 4: The dot product of two vectors are the multiplication of their length where one of the vectors is projected to the other.

2 Linear and circular motion

2.1 Linear motion

Linear motion is motion along a straight line, and can therefore be described mathematically using only one spatial dimension. It can be uniform, that is, with constant velocity (zero acceleration), or non-uniform, that is, with a variable velocity (non-zero acceleration). The motion of a particle (a point-like object) along the line can be described by its position x, which varies with t (time).

An example of linear motion is that of a ball thrown straight up and falling back straight down.

The average velocity v during a finite time span of a particle undergoing linear motion is equal to $\overline{v} = \sum x / \sum t$, where $\sum x$ is the total displacement and $\sum t$ denotes the time needed.

The instantaneous velocity of a particle in linear motion may be found by differentiating the position x with respect to the time variable t: $v = dx/dt$. The acceleration may be found by differentiating the velocity: $a = dv/dt$. By the fundamental theorem of calculus the converse is also true: to find the velocity when given the acceleration, simply integrate the acceleration with respect to time; to find displacement, simply integrate the velocity with respect to time.

This can be demonstrated graphically as Fig 5 demonstrates. The gradient of a line on the displacement time graph represents the velocity. The gradient of the velocity time graph gives the acceleration while the area under the velocity time graph gives the displacement. The area under an acceleration time graph gives the velocity.

The SI base unit of the velocity and the acceleration is m/s and m/s^2, respectively.

Also important to emphasize that the above formulae are valid only if the motion is linear i.e. the direction of the motion does not change. For motions where the direction is changing, the vector representation shall be used: $\mathbf{v} = d\mathbf{x}/dt$ and $\mathbf{a} = d\mathbf{v}/dt$.

2.2 Circular motion and rotation

Circular motion is movement along a circle: a circular path or a circular orbit. Circular motion can be interpreted for a mass point and a rigid body as well. Rotation is a rotary motion of a rigid body around an axis. (The connection between the rotation and the circular motion is, that each mass point of a rotating rigid body are performing a circular motion.)

Our planet, Earth has both kind of these motions: it orbits the Sun (circular motion) and rotates around its own axis (rotation). Orbiting causes

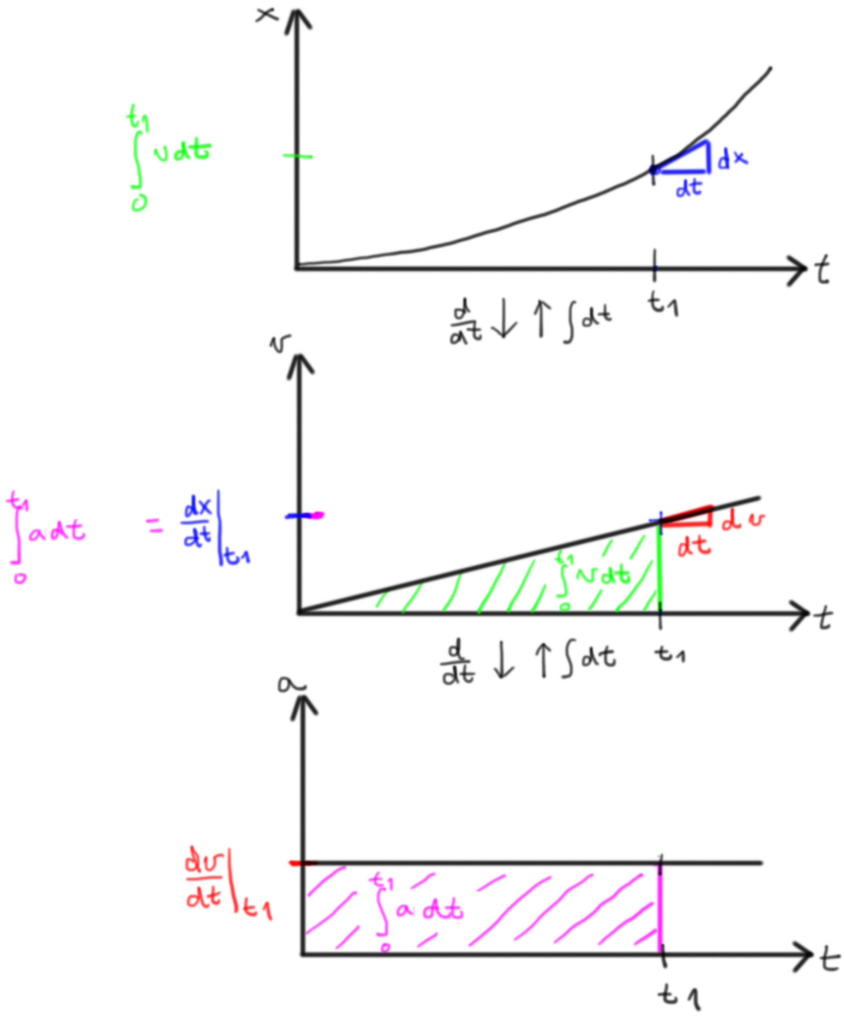


Figure 5: Graphical connection between the displacement, velocity, acceleration and their derivatives and integrates

the seasons and rotating causes the days and nights. Other examples of circular motion are: a stone which is tied to a rope and is being swung in circles (e.g. hammer throw), a racecar turning through a curve in a race track, an electron moving perpendicular to a uniform magnetic field, a gear turning inside a mechanism.

Circular motion can be uniform, that is, with constant angular rate of rotation, or non-uniform, that is, with a changing rate of rotation.

Circular motion is accelerated even if the angular rate of rotation is constant, because the object's velocity vector is constantly changing direction. Such change in direction of velocity involves acceleration of the moving ob-

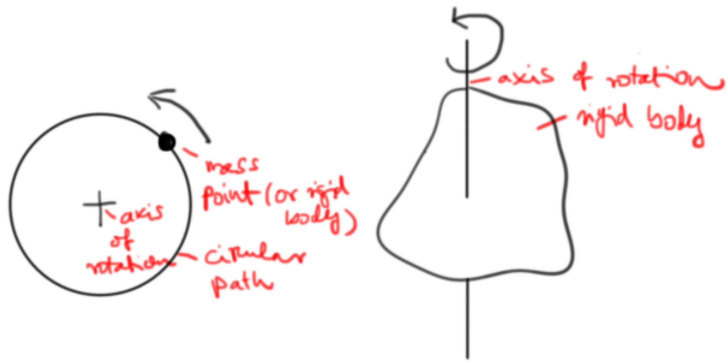


Figure 6: Difference between circular motion and rotation

ject by a centripetal force, which pulls the moving object towards the center of the circular orbit. This acceleration is known as centripetal acceleration (see its derivation in the next paragraph). Without this, the object would move in a straight line, according to Newton's laws of motion.

The velocity of the object traveling the circle is the circumference of the circle divided by the time for one complete rotation, $v[m/s]$:

$$v = \frac{2\pi R}{T} \tag{4}$$

The angular rate of rotation, also known as *angular velocity*, is the angle of a full circle in [radian] divided by the time for one complete rotation, $\omega\,[rad/s]$ is:

$$\omega = \frac{2\pi}{T} \tag{5}$$

From to above two equations one can obtain:

$$\omega = \frac{v}{R} \tag{6}$$

In mechanical engineering, the *revolution number* is often used, $n[1/min]$:

$$n = \frac{\omega}{2\pi} \times 60 \left[rpm = \frac{\text{number of rotations}}{\text{minute}} \right] \tag{7}$$

The angle θ swept out in a time t is

$$\theta = 2\pi \frac{t}{T} = \omega t. \tag{8}$$

The acceleration due to change in the direction of the velocity is found by analysing the change of the velocity vector in (small) time interval Δt. As $\omega = \text{const.}$, we have $|v_1| = |v_2| := v$. From the triangle in Fig. 7 we see that

$$\sin \frac{\Delta\varphi}{2} = \frac{\frac{\Delta v}{2}}{v} \tag{9}$$

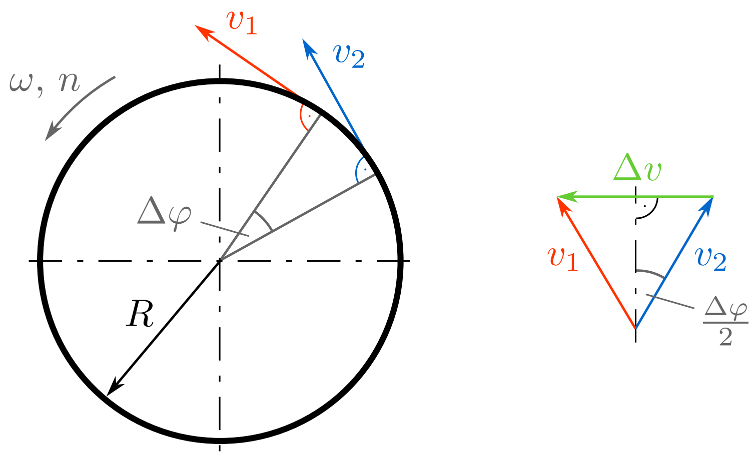


Figure 7: Circular motion, deriving centripetal acceleration

From this:

$$\Delta v = 2v \sin \frac{\Delta\varphi}{2} \tag{10}$$

Let $d\varphi$ denote a differential small angle change, then $\sin\alpha = \alpha$, so

$$dv = 2v\frac{d\varphi}{2} = vd\varphi = v\omega dt \tag{11}$$

The magnitude of centripetal acceleration a_{cp}, $[m/s^2]$ is

$$a = \frac{dv}{dt} = v\omega dtdt = R\omega\omega = R\omega^2 = \frac{v^2}{R} \tag{12}$$

The direction of the centripetal acceleration equals to the direction of the velocity change thus it points towards the center of rotation as shown in Fig. 8.

As it was mentioned earlier, the above centripetal acceleration is present even if the angular velocity is constant. If the angular velocity itself is varying in time, the *angular acceleration* $\varepsilon\,[rad/s^2]$ can be computed as

$$\varepsilon = \frac{\Delta\omega}{\Delta t}. \tag{13}$$

2.3 Newton's first law

Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.

This law states that if the resultant force (the vector sum of all forces acting on an object) is zero, then the velocity of the object is constant. Consequently:

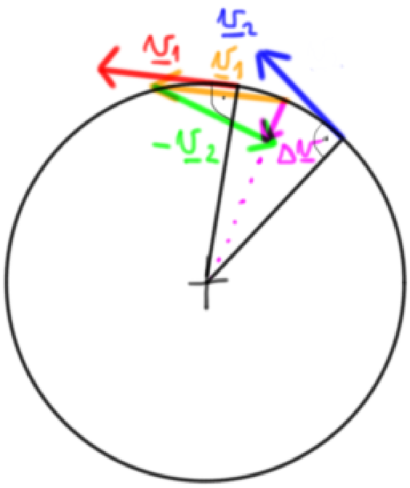


Figure 8: The direction of the centripetal acceleration is the direction of the velocity change Δv.

- An object that is at rest will stay at rest unless an unbalanced force acts upon it. E.g. You standing on Earth.
- An object that is in motion will not change its velocity unless an unbalanced force acts upon it. E.g. When a spacecraft left Earth far enough and reached its cruised speed it switches off its jets. It will keep the cruising speed since there is no friction or any other counterforce in the space. Another example is a car traveling with constant speed in a straight motorway (when the pulling forces and friction forces are equal and thus the resulting force is zero).

Newton placed the first law of motion to establish *frames of reference* for which the other laws are applicable. The first law of motion postulates the existence of at least one frame of reference called a Newtonian or inertial reference frame, relative to which the motion of a particle not subject to forces is a straight line at a constant speed. The frame of reference e.g. can be the Earth (for investigating motions on Earth or the coordinate system of our solar system.)

2.4 Newton's second law

The second law states that the net force on a particle is equal to the time rate of change of its linear momentum (p, defined as $p = mv$) in an inertial reference frame:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}, \tag{14}$$

where we assumed constant mass. Thus, the net force applied to a body produces a proportional acceleration. Note, that both F and a are vector quantities and they point in the same direction according to the above equation.

For circular motion or rotation of rigid bodies, we have to first define Torque. Torque is a vector quantity and defined as the vectorial product of a force and a leverage: $\mathbf{T} = \mathbf{F} \times \mathbf{r}$. The cross product was not defined in the introduction chapter of this lecture notes. In many cases the result of a cross product lies in a different plane than determined by the two vectors which the cross product is calculated from. We give You a graphical interpretation of the torque in text figure.

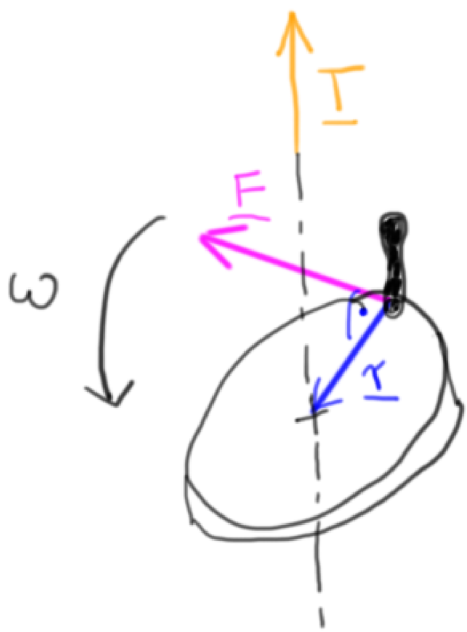


Figure 9: Torque causing rotation by acting a force on a lever

Calculating the magnitude of the torque is easy (despite the difficult vectorial calculation): $T = Fr$

Newton's second low for the circular and rotating motion is:

$$T = \theta\varepsilon, \tag{15}$$

with ε denotes angular acceleration and $\theta\,[kgm^2]$ is the *moment of inertia*. Some equations for the moment of inertia:

- a **mass point** rotating on a circle of radius r: $\theta = mr^2$
- a thin **ring** of radius r rotating around its own axis: $\theta = mr^2$

- a thin **disc** of radius r rotating around its own axis: $\theta = \frac{1}{2}mr^2$. See how to achieve this formula under the present list.
- a thin rod of mass m and length l, rotating around the axis which passes through its **center** and is perpendicular to the rod: $\theta = \frac{1}{12}ml^2$
- a thin rod of mass m and length l, rotating around the axis which passes through its **end** and is perpendicular to the rod: $\theta = \frac{1}{2}ml^2$
- a **solid ball** of mass m and radius r, rotating around an axis which passes through the center: $\theta = \frac{2}{5}mr^2$

Let us calculate the moment of inertia of a disc of height b, radius R and uniform density ρ at its own axis. We divide the radius into $N+1$ rings: $r_i = iR/N = i\Delta r$. The moment of inertia of the i^{th} ring is

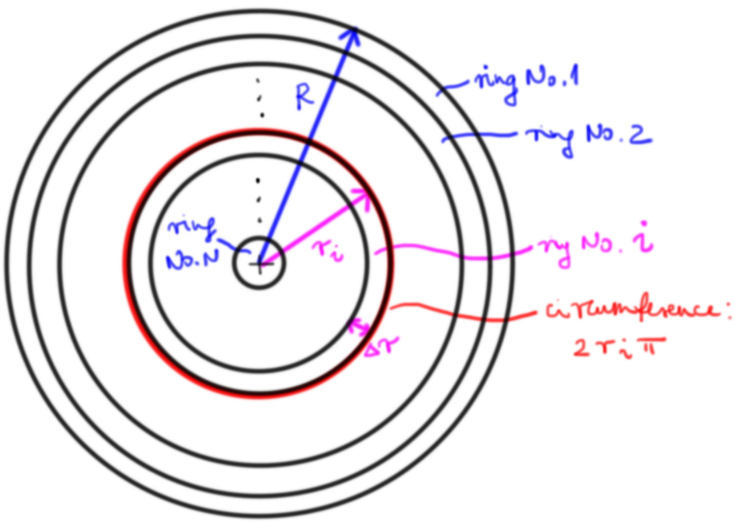


Figure 10: Braking down a disk into rings to calculate the moment of inertia

$$\theta_i = m_i r_i^2 = \underbrace{\underbrace{2r_i\pi}_{\text{circumference}} \quad \Delta r}_{\text{area}} \ b\rho \, r_i^2 = 2i\frac{R}{N}\pi\frac{R}{N}\,b\rho\left(i\frac{R}{N}\right)^2 \tag{16}$$

By summing up these rings we obtain

$$\begin{aligned}\theta_{\text{disc}} &= \lim_{N\to\infty}\sum_{i=1}^{N}\theta_i = \lim_{N\to\infty} 2\left(\frac{R}{N}\right)^4 \pi b\rho \sum_{i=1}^{N} i^3 \\ &= \lim_{N\to\infty} 2\left(\frac{R}{N}\right)^4 \pi b\rho\frac{1}{4}N^2(1+N^2) = 2R^4\pi b\rho\frac{1}{4} = \frac{1}{2}mR^2 \end{aligned}\tag{17}$$

Note that the same result could be obtain by formal integration of the moment of inertia of the rings from 0 to R.

The *parallel axis theorem* (or Huygens-Steiner theorem) can be used to determine the moment of inertia of a rigid body about *any axis*, given the moment of inertia of the object about the parallel axis through the object's centre of mass and the perpendicular distance (r) between the axes. The moment of inertia about the new axis z is given by:

$$\theta_z = \theta_{cm} + mr^2 \tag{18}$$

where θ_{cm} is the moment of inertia of the object about an axis passing through its centre of mass, m is the object's mass and r is the perpendicular distance between the two axes. For example, let us compute the moment of inertia of a thin rod rotating around the axis which passes through its **end**:

$$\theta_{end} = \theta_{cm} + m\left(\frac{l}{2}\right)^2 = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 = \frac{1}{3}ml^2. \tag{19}$$

Finally, the moment of inertia of an object can be computed simply as the sum of moments of inertia of its "building" objects.

2.5 Newton's third law

If a body acts wth force to a second body, the second body acts with a counterforce which is

- same in magnitude
- opposite in direction

2.6 Work

In physics, *mechanical work* is the amount of energy transferred by a force acting through a distance. In the simplest case, *if the force and the displacement are parallel and constant*, we have

$$W = Fs. \tag{20}$$

It is a scalar quantity, with SI units of *joules*. If the direction of the force and the displacement do not coincide (e.g. when pulling a bob up to a hill) - but they are still constant - one has to take the scalar product of force and displacement (as it was defined in Section 1.3):

$$W = \mathbf{F}\cdot\mathbf{s} = |\mathbf{F}|\,|\mathbf{s}|\cos\theta = Fs\cos\theta, \tag{21}$$

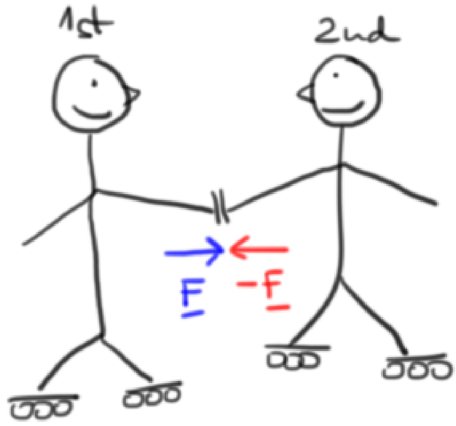


Figure 11: Because of the counterforce both person will start to move on their rolling skates. Counterforce is same in magnitude and opposite in direction.

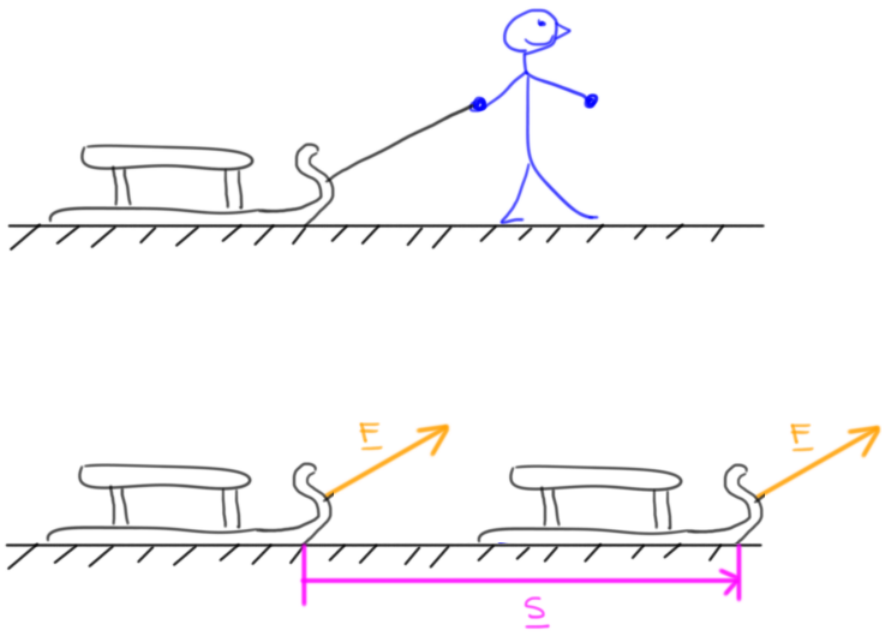


Figure 12: An easy example where force and displacement is not parallel: a child pulling a sleight

where θ is the angle between the force and the displacement vector. An easy example of this case is depicted in Fig. 12.

The unit of the work is $[N] \cdot [m] = [Nm] = [J]$, Joule. It can be expressed by base SI units as $[Nm] = [kgm/s^2 \cdot m] = kgm^2/s^2$.

In situations where the force changes over time, or the path deviates from a straight line, equation (20) is not generally applicable although it is possible to divide the motion into small steps, such that the force and motion are well approximated as being constant for each step, and then to express the overall work as the sum over these steps. Mathematically, the calculation of the work needs the evaluation of the following line integral:

$$W_C = \int_C \mathbf{F} \cdot d\mathbf{s}, \tag{22}$$

where C is the path or curve traversed by the object; $\mathbf{F}$ is the force vector; and $\mathbf{s}$ is the position vector. Note that the result of the above integral depends on the path and not only from the endpoints. This is typical for systems in which losses (e.g. friction) are present (similarly as the actual fare of a taxi from point A to B depends heavily on the route the driver chooses).

For circular motion or rotation, the work is defined as

$$W = T\varphi, \tag{23}$$

, where T is the torque and φ is the angle rotated. Note that the unit is (of course) the same as for linear motion: $[Nm] = [J]$.

2.7 Energy

Energy is a quantity that is often understood as the ability to perform work. This quantity can be assigned to any particle, object, or system of objects as a consequence of its physical state.

Energy is a scalar physical quantity. It is In the International System of Units (SI), energy is measured in joules just like work and in case of certain conditions and with known limitations work end energy can transfer to each other. In some fields other units such as kilowatt-hours and kilocalories are also used. Different forms of energy include kinetic, potential, thermal, gravitational, sound, elastic and electromagnetic energy.

Any form of energy can be transformed into another form. When energy is in a form other than thermal energy, it may be transformed to other type of energy, however, during this conversion a portion of energy is usually lost because of losses such as friction, imperfect heat isolation, etc.

In mechanical engineering, we are mostly concerned with the following types of energy (examples, there are much more that will be detailed later in the lecture notes):

- potential energy: $E_p = mgh$
- kinetic energy for linear motion: $E_k = \frac{1}{2}mv^2$

- kinetic energy for circular / rotating motion: $E_k = \frac{1}{2}\theta\omega^2$
- internal energy: $E_t = c_p m T$.

Although the total energy of an *isolated* system does not change with time[1], its value may depend on the frame of reference. For example, a seated passenger in a moving airplane has zero kinetic energy relative to the airplane, but non-zero kinetic energy (and higher total energy) relative to the Earth.

A *closed* system interacts with its surrounding with mechanical work (W) and heat transfer (Q). Due to this interaction, the energy of the system changes:

$$\Delta E = W + Q, \tag{24}$$

where work is positive if the system's energy increases (e.g. by lifting objects their potential energy increases) and heat transfer is positive if the temperature of the system increases. We must note that the laws of thermodynamics details these transformations and also describe significant limitations e.g. if You push a sleight on the floor away, the floor will heat up but if You cool the floor down it will not cause the sleight to slide back to its original place.

2.8 Power

Power is the rate at which work is performed or energy is converted. If ΔW is the amount of work performed during a period of time of duration Δt, the average power $\overline{P}$ over that period is given by

$$\overline{P} = \frac{\Delta W}{\Delta t}. \tag{25}$$

A very straightforward example for power is given as follows. Shooting a gun or burning a candle are both release roughly 0.1 MJ energy. But in the first case, this energy is released in 0.01s and in the latter one it lasts for 4 hours. This is visible from the calculated power, which is 10 MJ for the gunshooting and 6.9 W for the candle.

The average power is often simply called "power" when the context makes it clear. The instantaneous power is then the limiting value of the

[1]There is a fact, or if you wish, a law, governing all natural phenomena that are known to date. There is no known exception to this law—it is exact so far as we know. The law is called the conservation of energy. It states that there is a certain quantity, which we call energy, that does not change in manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same. —The Feynman Lectures on Physics

average power as the time interval Δt approaches zero. In the case of constant power P, the amount of work performed during a period of duration T is $W = PT$. Depending on the actual machine, we have

mechanical (linear motion) power: $P = Fv \left[N\frac{m}{s}\right]$

mechanical (rotation) power: $P = M\omega \left[Nm\frac{rad}{s}\right]$

electrical power: $P = UI\ [VA]$

hydraulic power: $P = Q\Delta p \left[\frac{m^3}{s}Pa\right]$

The dimension of power is energy divided by time J/s. The SI unit of power is the watt (W), which is equal to one joule per second. A common non-SI unit of power is horsepower (hp), $1hp = 0.73549875kW$.

2.9 Comparison of linear and circular (rotating) motion

As a summary, we collect some physical quantities for linear and circular motion as a comparison.

Quantity	Linear motion	Circular motion / rotation
displacement / angle	$s = v_0t + a/2 \cdot t^2[m]$	$\varphi = \omega_0 t + \varepsilon/2 \cdot t^2[rad]$
velocity / angular velocity	$v = s/t[m/s]$	$\omega = \Delta\varphi/\Delta t[rad/s, 1/s]$
acceleration / angular acceleration	$a = \Delta v/\Delta t[m/s^2]$	$\varepsilon = \Delta\omega/\Delta t[rad/s^2, 1/s^2]$
centripetal acceleration	-	$a = v^2/r = \omega^2 r[m/s^2]$
force / torque (N's 2^{nd} law)	$F = ma[N]$	$T = \theta\varepsilon$
work	$W = Fs[J]$	$W = T\varphi[J]$
power	$P = Fv[W]$	$W = T\omega[W]$

Table 3: Linear and circular motion

2.10 Problems

Problem 2.1 An internal combustion engine has a torque of $500Nm$ when the revolution number is $2500rpm$.

- Calculate the mechanical power of the engine. ($130.9kW$).

Problem 2.2 Calculate the total electrical power of a vacuum cleaner if the available voltage is $240V$ and the electrical current is $9A$. ($P = 2160VA$)

Problem 2.3 We throw a ball upwards vertically with an initial velocity of 18 m/s.

- Calculate the maximum altitude of the ball. (16.51 m).
- How much time does it take for the ball to reach the ground again? (3.67 s)
- Calculate the mechanical power of the ball at the moment of landing if it hits the ground with a force of 3 N. (54 W)

Problem 2.4 Calculate the hydraulic power of a volumetric pump if the change of pressure is $200bar$ and the volumetric flow rate is $2.5\frac{l}{s}$. ($50kW$)

Problem 2.5 We drive by car for 4 hours, after which we refuel $32l$ of gasoline. The car has a 55kW motor (75hp) and it can be assumed that during the journey this was the useful power. The heating value of gasoline is $35\,MJ/l$.

- Calculate the useful work ($W_u = 220kWh = 792MJ$), input energy ($E_i = 1120MJ$) and efficiency ($\eta = 70.7\%$).

Problem 2.6 A truck moves on a highway with a constant speed of $90km/h$. A sudden traffic jam forces the driver to stop the vehicle. The weight of the truck is $9t$.

- How many litres of oil do we need to cool the brake pads if its maximum allowed temperature is $40°C$? The specific heat capacity of the oil is $2\frac{kJ}{kg°C}$, its density is $0.9\frac{g}{cm^3}$. ($V = 78.12l$)

Problem 2.7 A $210MW$ coal plant consumes $4100t$ of coal per day. The heating value of lignite is $17MJ/kg$.

- Calculate the efficiency of the plant ($\eta = 26\%$).

Problem 2.8 A rotating wheel of a vehicle is stopped by two brakes as shown in Figure 13. The friction coefficient is $\mu = 0.13$, the diameter of the wheel is $910mm$, the initial velocity of the vehicle was $65km/h$. The pushing force is $F = 6000N$.

- Calculate the friction force.
  $F_f = \mu F = 0.78kN$.
- Calculate the (overall) braking torque acting on the wheel.
  $M_f = 2 \times F_f \frac{D}{2} = 0.7098kNm$
- Calculate the power of braking at the start of the breaking.
  $P = M_f \omega_0 = M_f \frac{2v_0}{D} = 28.17kW$
- Assuming constant torque and linearly decreasing velocity (i.e. constant deceleration), compute the time needed to stop a 20t vehicle with six braked weels.
  The initial kinetic energy of the vehicle is $E_k = \frac{1}{2}mv_0^2 = 3.26MJ$.
  The overall work done by the six brakes $W = 6 \times M_f T \frac{\omega}{2}$ (T is yet unknown).
  The initial kinetic energy is fully dissipated by the braking work: $E_k = W \rightarrow T = 38.6s$

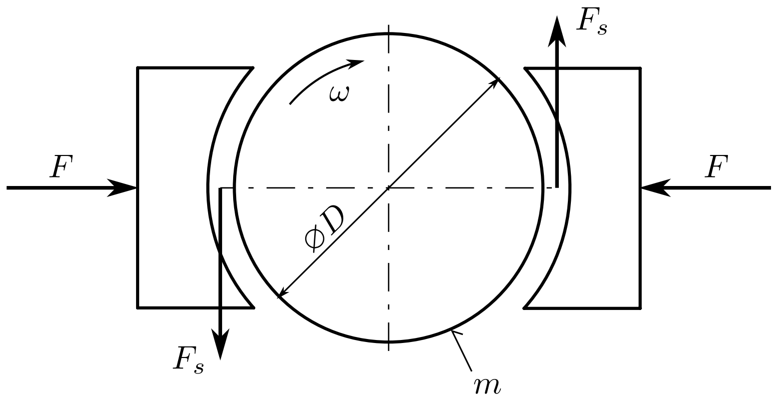


Figure 13: Braking a rotating wheel.

Problem 2.9 We ride a bicycle at a speed of $30km/h$ as shown in Figure 14. The diameter of the front wheel is $700mm$, its weight is $600g$. For speed measurement a small magnet is attached to the wheel $200mm$ away from the axis. The point-like magnet's weight is $50g$.

- Calculate the maximum acceleration of the axis of the front wheel.
  $a = \frac{v^2}{R} = 198.4\frac{m}{s^2}$
- Calculate the time needed for the magnet to take a whole turn. How many turns does it take in a minute?
  $T = \frac{2\pi R}{v} = 0.26s$
  $n = \frac{1}{T} = 230.8\frac{1}{min}$
- Calculate the moment of inertia for the magnet.
  $\theta = m_m \times r_m^2 = 0.002kgm^2$
- Calculate the angular acceleration of the front tire if we stop the bike in 10 seconds.
  $\omega = \frac{v}{R} = 23.81\frac{rad}{s}$
  $\varepsilon = \frac{\Delta\omega}{\Delta t} = -2.381\frac{rad}{s^2}$

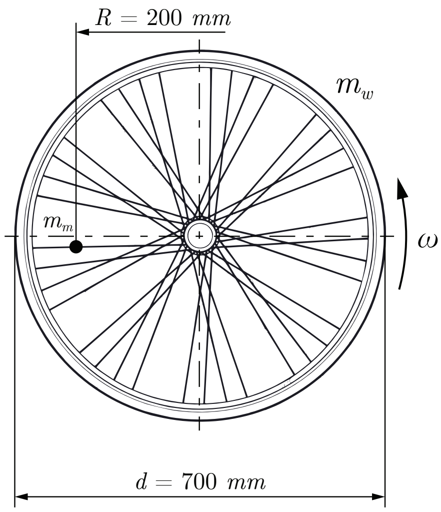


Figure 14: Front wheel of the bike.

3 Steady-state operation of machines

3.1 Springs

A spring is an elastic object that stores mechanical energy; that is, when force is applied the spring length decreases (the spring compresses). Once the force is lifted, the spring will expand to its original length.

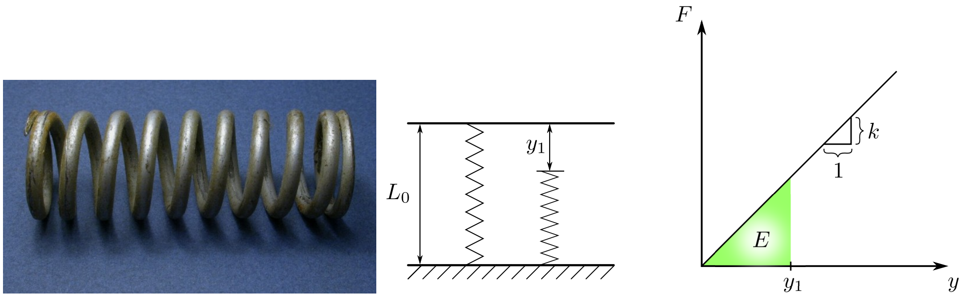


Figure 15: Left: helical spring. Right: energy content of compressed spring.

As long as not stretched or compressed beyond their elastic limit (i.e. the spring is not damaged), the force is linearly proportional to the distance from its equilibrium length:

$$F = kx \tag{26}$$

where $x\,(m)$ is the displacement, $F\,(N)$ is the resulting force and $k\,(N/m)$ is the spring stiffness which depends on the spring's material and construction. Note that the notation of the displacement can vary, e.g. x, y, l, s. Sometimes the spring constant is used for calculations which is the reciprocal of the stiffness. One can be sure of the correct use of sprint stiffness / constant by analysing the unit equality of the calculations.

Assuming that we compress the spring from the original (equilibrium, free) length to y_1 (that is, the original length was L_0 and the compressed length is $L_0 - y_0$, see Fig.15), the energy stores is

$$W = \int_0^{y_1} F(y)\,dy = \int_0^{y_1} ky\,dy = \frac{k}{2}y_1^2. \tag{27}$$

When connected in parallel – see Figure 16 right panel – the displacement is the same while the spring forces add up:

$$F = F_1 + F_2 = k_1 y + k_2 y = \underbrace{(k_1 + k_2)}_{k_{red.,parallel}}\,y, \tag{28}$$

hence, one can define the "reduced" spring stiffness $k_{red.,parallel}$, which is one single spring that behaves the same way as the two springs connected in parallel.

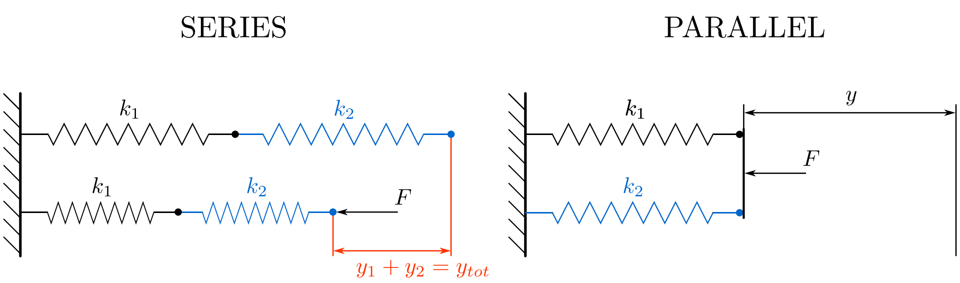


Figure 16: Springs, connected in series (left) and parallel (right).

Similarly, one can analyse the case when the springs are connected in series as in the left panel of Figure 16. We have

$$y = y_1 + y_2 = \frac{F}{k_1} + \frac{F}{k_2} = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \quad \rightarrow \quad \underbrace{\frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}}_{k_{red,series}} y = F. \tag{29}$$

The general case for N springs gives

- spring connected parallel: $k_{red.} = \sum_i k_i$ and
- spring connected in series: $k_{red.} = \left(\sum_i \frac{1}{k_i}\right)^{-1}$.

3.2 The static and kinetic friction force due to dry friction

Dry friction resists relative lateral motion of two solid surfaces in contact. The two regimes of dry friction are *static friction* between non-moving surfaces, and *kinetic friction* (sometimes called sliding friction or dynamic friction) between moving surfaces. Coulomb friction is an approximate model used to calculate the force of dry friction:

$$|F_f| \leq \mu N. \tag{30}$$

where

- F_f is the force exerted by friction (in the case of equality, the maximum possible magnitude of this force).
- μ is the coefficient of friction, which is an empirical property of the contacting materials (see typical values in Table 4),
- N is the normal force exerted between the surfaces.

Further characterisation of the static and kinetic (sliding) friction follows.

Materials		Dry and clean	Lubricated
Aluminum	Steel	0.61	
Copper	Steel	0.53	
Brass	Steel	0.51	
Cast iron	Copper	1.05	
Cast iron	Zinc	0.85	
Concrete (wet)	Rubber	0.30	
Concrete (dry)	Rubber	1.0	
Concrete	Wood	0.62	
Copper	Glass	0.68	
Glass	Glass	0.94	
Metal	Wood	0.2-0.6	0.2
Polythene	Steel	0.2	0.2
Steel	Steel	0.80	0.16
Steel	Teflon	0.04	0.04
Teflon	Teflon	0.04	0.04
Wood	Wood	0.25-0.5	0.2

Table 4: Approximate coefficients of friction

- Static friction: The Coulomb friction may take any value from zero up to μN, and the direction of the frictional force against a surface is opposite to the motion that surface would experience in the absence of friction. Thus, in the static case, the frictional force is exactly what it must be in order to prevent motion between the surfaces; it balances the net force tending to cause such motion. In this case, rather than providing an estimate of the actual frictional force, the Coulomb approximation provides a threshold value for this force, above which motion would commence. This maximum force equals to the traction force.

- Kinetic friction: The force of friction is always exerted in a direction that opposes movement (for kinetic friction) or potential movement (for static friction) between the two surfaces. For example, a curling stone sliding along the ice experiences a kinetic friction force slowing it down. For an example of potential movement, the driven wheels of an accelerating car experience a frictional force pointing forward; if they did not, the wheels would spin, and the rubber would slide backwards along the pavement. Note that it is not the direction of movement of the vehicle they oppose, it is the direction of (potential) sliding between tire and road.

3.3 About the direction of the friction force

In the case of kinetic friction, the direction of the friction force may or may not match the direction of motion: a block sliding atop a table with rectilinear motion is subject to friction directed along the line of motion; an automobile making a turn is subject to friction acting perpendicular to the line of motion (in which case it is said to be 'normal' to it). The direction of the static friction force can be visualized as directly opposed to the force that would otherwise cause motion, were it not for the static friction preventing motion. In this case, the friction force exactly cancels the applied force, so the net force given by the vector sum, equals zero. It is important to note that in all cases, Newton's laws of motion hold.

Another use case is a bike drive what is visualised in Fig.17, let us investigate the static friction on both wheels:

- Friction force on the front wheel. The front wheel is connected to the rest of the bicycle by a rod passing through its centre (axle). The torque on the wheel about its centre by the force coming from the rest of the bicycle is zero. Thus, pedalling can give linear velocity (only) to the front wheel but cannot rotate it. As there is only one (horizontal) force acting on the wheel and it is pointing forward, the friction force must point backwards to balance this force. Hence, the friction force points backwards on the front wheel.

- Friction force on the rear wheel. The rear wheel is connected to rest of the bicycle by a rod passing through its centre and a chain connected to the pedals. Pressing the pedal increases tension in the upper portion of the chain. This tension gives rise to clockwise torque and wheel starts rotating in clockwise direction. To ensure the torque equilibrium, the friction force must point forward to generate an anti-clockwise torque (balancing the torque of the pedaling). Hence, the friction force points forward on the rear wheel. If pedalling is stopped, the direction of friction force on the rear wheel gets reversed.

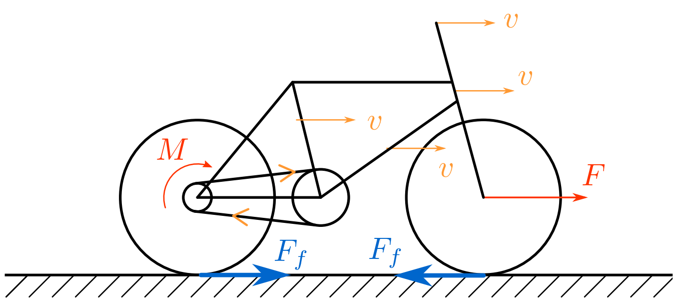


Figure 17: Friction force on a bike.

3.4 Rolling resistance

Rolling resistance, sometimes called rolling friction or rolling drag, is the resistance that occurs when a round object such as a ball or tire rolls on a flat surface, in steady velocity straight line motion. It is caused mainly by the deformation of the object, the deformation of the surface, or both. (Additional contributing factors include wheel radius, forward speed, surface adhesion, and relative micro-sliding between the surfaces of contact.) It depends very much on the material of the wheel or tire and the sort of ground.

For example, rubber will give a bigger rolling resistance than steel. Also, sand on the ground will give more rolling resistance than concrete. A moving wheeled vehicle will gradually slow down due to rolling resistance including that of the bearings, but a train car with steel wheels running on steel rails will roll farther than a bus of the same mass with rubber tires running on asphalt. The coefficient of rolling resistance is generally much smaller for tires or balls than the coefficient of sliding friction.

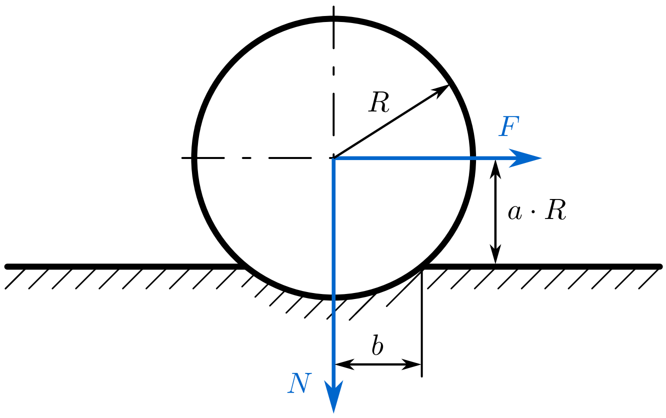


Figure 18: Hard wheel rolling on and deforming a soft surface.

The force of rolling resistance (see Fig.18) can also be calculated from the torque equilibrium:

$$FaR = Nb \tag{31}$$

$\rightarrow$

$$F = \frac{Nb}{aR} = C_{rr}N \tag{32}$$

where

- F is the rolling resistance force,
- aR is the distance of the wheel axis and the not deformed road surface,

- b is the rolling resistance coefficient or coefficient of rolling friction with dimension of length,
- $C_{rr} = b/aR$ is the $c$oefficient of $r$olling $r$esistance (dimensionless number), and
- N is the normal force.

Note that the above derivation applies only for the cases when the surface is deformable and the wheel considered to be rigid. Different formulae apply if the surface is considered rigid and the wheel is deformable or if both are deformable.

C_{rr}	b	Description
0.0002...0.0010	0.5 mm	Railroad steel wheel on steel rail
	0.1mm	Hardened steel ball bearings on steel
0.0025		Special Michelin solar car/eco-marathon tires
0.005		Tram rails standard dirty with straights and curves[citation needed]
0.0055		Typical BMX bicycle tires used for solar cars
0.0062...0.015		Car tire measurements
0.010...0.015		Ordinary car tires on concrete
0.3		Ordinary car tires on sand

Table 5: Approximate coefficients of rolling resistance

In usual cases, the normal force on a single tire will be the mass of the object that the tires are supporting divided by the number of wheels, plus the mass of the wheel, times the gravitational acceleration. In other words, the normal force is equal to the weight of the object being supported, if the wheel is on a horizontal surface.

3.5 Statics of objects on inclined planes (decomposing forces)

To calculate the forces on an object placed on a slope (inclined plane), consider the forces acting on it:

- the normal force (N) exerted on the body by the plane due to the force of gravity,
- the force due to gravity (mg, acting vertically downwards) and
- the frictional force (F_f) acting parallel to the plane.

It is important to emphasise that the following derivation is valid only for a specific case (depicted in Fig.19) of possible object-slope scenarios: the body can be pulled / pushed, β can positive and negative, we can go uphill or downhill, etc. For each of this cases a similar derivation can be done, and a different end formula will be obtained.

We can decompose the gravitational force into two vectors, one perpendicular to the plane and one parallel to the plane. Since there is zero acceleration perpendicular to the plane, the component of the gravitational force in this direction ($mg\cos\alpha$) must be equal and opposite to normal force exerted by the plane, N plus the normal component of the force F:

$$mg\cos\alpha = F\sin\beta + N. \tag{33}$$

The remaining component of the gravitational force parallel to the surface ($mg\sin\alpha$) plus the friction force equals the the "pulling" force F_t:

$$mg\sin\alpha + F_f = F\cos\beta \tag{34}$$

By definition, the friction force is

$$F_f = \mu N. \tag{35}$$

Combining these three equations, we arrive at the following equation for the traction force F:

$$F = mg\frac{\sin\alpha + \mu\cos\alpha}{\cos\beta + \mu\sin\beta}. \tag{36}$$

Let us describe the three different cases while varying the force:

- If the actual traction force is smaller than F (but positive), the object is at rest. The value of the friction force is such that the body stays at rest: $F_f = mg\sin\alpha$ and points upwards to balance the tangential component of G.
- If the traction force is exactly F, the body is either at rest or moves with constant *arbitrary* velocity.

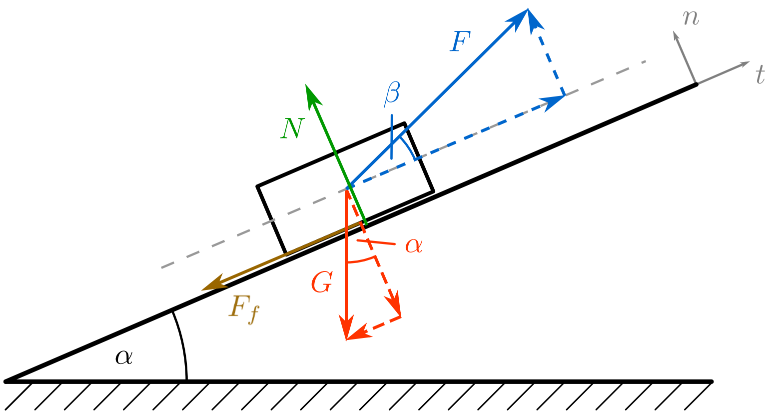


Figure 19: Object on an inclined plane.

- Finally, for traction force values beyond F the object experiences a constant acceleration.

One might want to compute the β^* angle for which the smallest F force is needed to move the object:

$$0 = \frac{dF}{d\beta} = \underbrace{mg\left(\sin\alpha + \mu\cos\alpha\right)}_{\text{const.}} \frac{d}{d\beta}\left(\cos\beta + \mu\sin\beta\right)^{-1}$$
$$= \text{const.}(-1)\frac{-\sin\beta^* + \mu\cos\beta^*}{(\cos\beta^* + \mu\sin\beta^*)^2} \quad \rightarrow \quad \beta^* = \arctan\mu \tag{37}$$

Note that our equations are valid only if $N > 0$, i.e.

$$G\cos\alpha \geq F_{max}\sin\beta \tag{38}$$

Finally, let us define the efficiency of the traction. The useful work is the vertical displacement $\Delta h = L\sin\alpha$ while the input work is the work done by tangential component of the force, i.e. F_t:

$$\eta = \frac{\Delta E_{pot}}{W_{F_t}} = \frac{mg\Delta h}{F_t L} = \frac{mgL\sin\alpha}{mgL\sin\alpha + F_f L}, \tag{39}$$

whose maximum value occurs if $F_f = 0$, i.e. when the friction is zero which can be zero only if the normal component vanishes. Hence the maximum-efficiency point corresponds to the point what is described by (38).

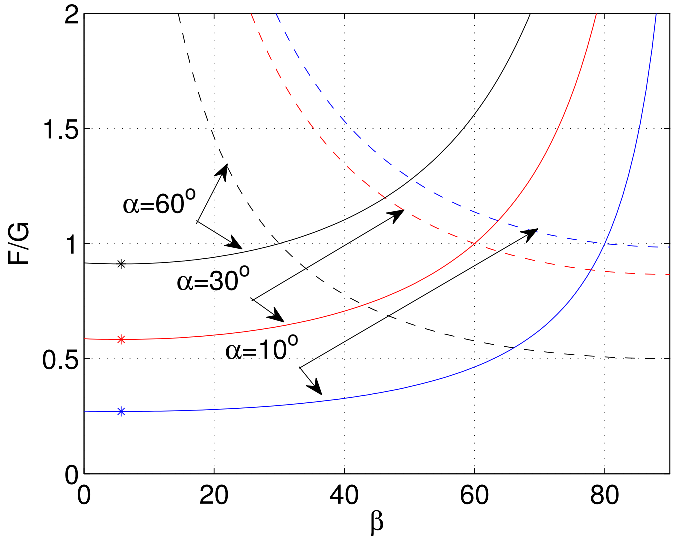


Figure 20: Solid line: the relative traction force needed as a function of β, see (36). Dashed line: the lift-up force given by (38). The friction coefficient is $\mu = 0.1$.

3.6 Pulley

3.6.1 Pulley without friction

A pulley, also called a sheave or a drum, is a mechanism composed of a wheel on an axle or shaft that may have a groove between two flanges around its circumference. A rope, cable, belt, or chain usually runs over the wheel and inside the groove, if present. Pulleys are used to change the direction or the magnitude of an applied force, transmit rotational motion. Two or more pulleys together are called a block and tackle.

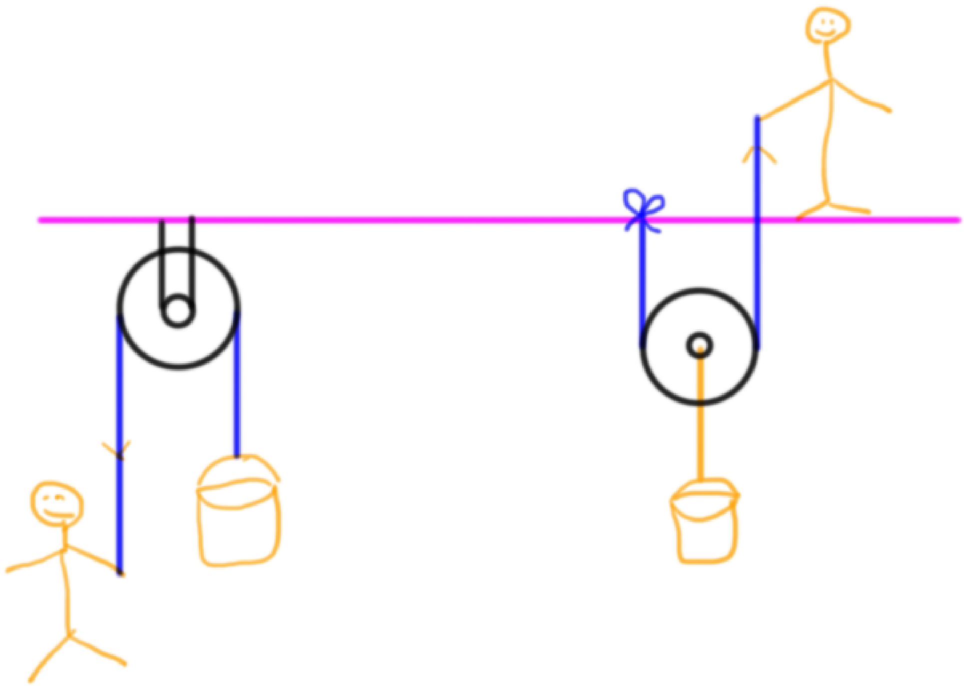


Figure 21: Fixed and moving pulleys. The fixed pulley just changing the direction of the necessary pulling force while the moving pulley also reduces the necessary pulling force to the half.

The different types of pulley systems are:

Fixed A fixed pulley has a fixed axle. That is, the axle is "fixed" or anchored in place. A fixed pulley is used to change the direction of the force on a rope (called a belt).

Movable A movable pulley has a free axle. That is, the axle is "free" to move in space. A movable pulley is used to multiply forces.

Compound A compound pulley is a combination of a fixed and a movable pulley system. The *block and tackle* is a type of compound pulley where several pulleys are mounted on each axle, further increasing the mechanical advantage.

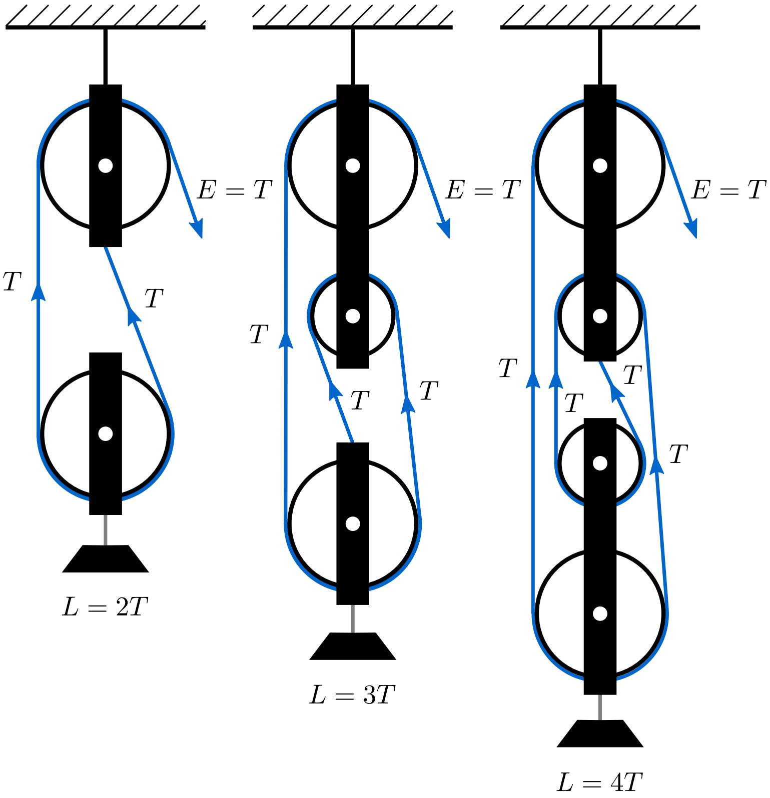


Figure 22: Compound pulley systems. L = load force, T = force in one branch of the rope.

The movable pulley depicted in the right side of Fig. 21 and the compound pulley shown in the left side of Fig. 22 are both reducing the necessary pulling force (ore rope force, T) to the half of the load force L. It is easy to admit by writing the force equilibrium of $2T = L$. Based on the energy conservation (W=Fs constant), the distance which the end of the rope is pulled along shall be twice as the distance of the load traveled. So a moving pulley decreases the necessary load to the half by the cost of the distance will take twice as long.

It is important to note that as long as the friction due to sliding friction in the system where cable meets pulley and in the rotational mechanism of each pulley is neglected, the change in the potential energy of the weight $G\Delta h$ and the lifting work $F\Delta s$ are equal.

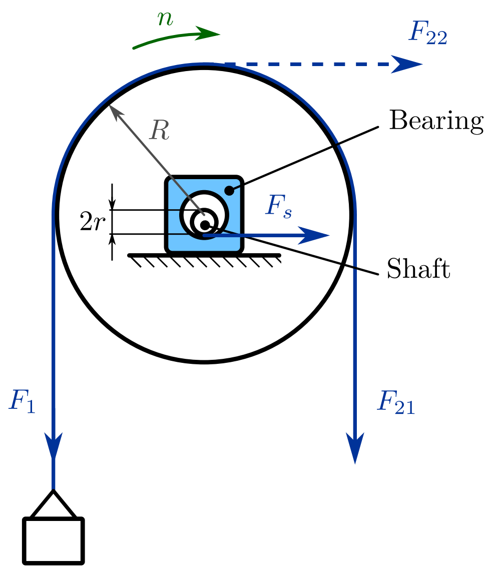


Figure 23: Pulley with friction

3.6.2 Pulley with friction

Figure 23 depicts a pulley with friction between the bearing and the shaft. F_1 represents the load force (in this case, one lifts a mass) while F_{21} and F_{22} shows two possible pulling directions. The friction force is

$$F_s = \mu N, \quad \text{where} \quad N = \begin{cases} F_1 + F_{21} & \text{for vertical arrangement, and} \\ \sqrt{F_1^2 + F_{22}^2} & \text{for horizontal arrangement.} \end{cases} \tag{40}$$

The torque equilibrium for both cases is given by

$$F_{2x} R = F_1 R + \mu N r. \tag{41}$$

In the case of horizontal arrangement, we have

$$F_{22} R = F_1 R + \mu \sqrt{F_1^2 + F_{22}^2}\, r \;\rightarrow\; F_{22} = F_1 \frac{1 + \delta\sqrt{2 - \delta^2}}{1 - \delta^2} \text{ with } \delta = \mu \frac{r}{R}. \tag{42}$$

Similarly for the vertical case, we have

$$F_{21} R = F_1 R + \mu (F_1 + F_{21}) r \;\rightarrow\; F_{21} = F_1 \frac{1 + \delta}{1 - \delta} \text{ with } \delta = \mu \frac{r}{R}. \tag{43}$$

3.7 Gear drive, friction drive and belt drive

3.7.1 Gear drive

A gear is a rotating machine part having cut teeth which mesh with another toothed part to transmit torque. Geared devices can change the speed, torque, and direction of a power source. Gears almost always produce a change in torque, creating a mechanical advantage, through their gear ratio, and thus may be considered a simple machine. An advantage of gears is that the teeth of a gear prevent slip.

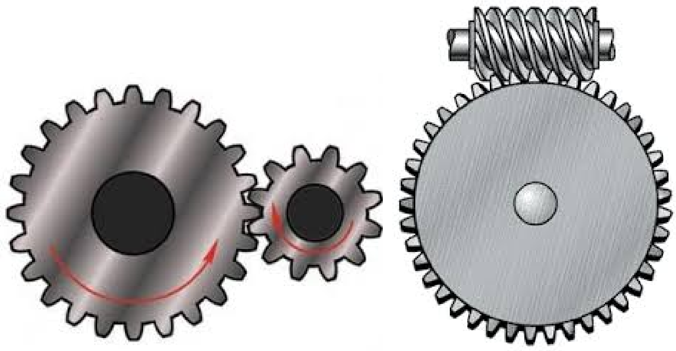


Figure 24: (left) gear drive (right) worm drive.

The circumferential velocity components are equal that is

$$v_1 = v_2 \quad \to \quad r_1\omega_1 = r_2\omega_2 \quad \to \quad \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{r_2}{r_1} = \frac{z_2}{z_1}, \tag{44}$$

meaning that the change in the revolution number will be proportional to the diameters. z denotes the number of teeth. Similarly, the driving forces are equal, hence

$$F_1 = F_2 \quad \to \quad \frac{M_1}{r_1} = \frac{M_2}{r_2} \quad \to \quad \frac{M_1}{M_2} = \frac{r_1}{r_2} = \frac{z_1}{z_2}. \tag{45}$$

Hence, gears change the revolution number and the torque in the opposite direction: if revolution number increases, torque will decrease and vica versa. The *gear ratio* of the transmission is given defined as

$$i = \frac{n_1}{n_2} = \frac{\omega_1}{\omega_2} = \frac{z_2}{z_1}. \tag{46}$$

3.7.2 Friction drive

The *friction drive* (see the left-hand side of Figure 25) or friction engine is a type of transmission that, instead of a chain and sprockets, uses two

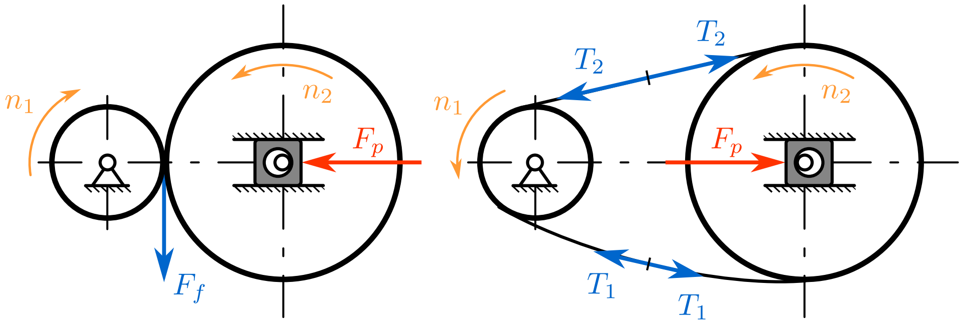


Figure 25: Friction drive (left) and belt drive (right).

wheels in the transmission to transfer power to the driving wheels. This kind of transmission is often used on scooters, mainly go-peds, in place of a chain.

The friction force transmitting the power is

$$F_f = F_1 = \mu F_{cl}, \tag{47}$$

where F_{cl} is the clamping force, i.e. the force pushing the two gears against each other.

For friction drive the same rules apply as for gear drives (44)-(46) with the difference that slip can appear which will be detailed in Section 3.7.4.

3.7.3 Belt drive

The *belt drive* (see the right-hand side of Figure 25) uses a belt, i.e. a loop of flexible material used to link two or more rotating shafts mechanically. Belts are looped over pulleys. In a two pulley system, the belt can either drive the pulleys in the same direction, or the belt may be crossed, so that the direction of the shafts is opposite. As a source of motion, a conveyor belt is one application where the belt is adapted to continually carry a load between two points.

Belt friction is a physical property observed from the forces acting on a belt wrapped around a pulley, when one end is being pulled. The equation used to model belt friction is, assuming the belt has no mass and its material is a fixed composition.

$$T_2 = T_1 e^{\mu_s \beta}, \tag{48}$$

where T_2 is the tension of the pulling side, which is typically the greater force, T_1 is the tension of the resisting side, μ_s is the static friction coefficient, which has no units, and β is the angle, in radians formed by the first and last spots the belt touches the pulley, with the vertex at the center of the pulley.

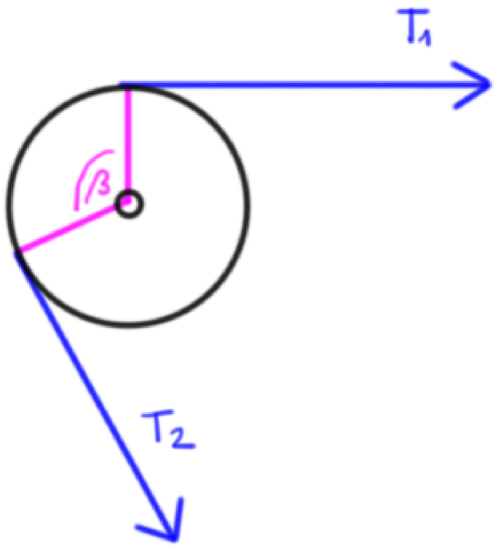


Figure 26: Forces on a belt drive

The tension on the pulling side has the ability to increase exponentially if the size of the angle increases (e.g. it is wrapped around the pulley segment numerous times) and as the coefficient of friction grows. The force needed to be applied to the shaft is

$$F_p = T_1 + T_2 = T_1 \left(1 + e^{\mu_s \beta}\right), \tag{49}$$

while the friction force transferring the driving torque is

$$\begin{aligned} F_f = \frac{M_1}{R_1} = T_1 - T_2 = \frac{(T_1 - T_2)(T_1 + T_2)}{T_1 + T_2} = \frac{T_1^2 \left(1 - \frac{T_2^2}{T_1^2}\right)}{T_1 \left(1 + e^{\mu_s \beta}\right)} \\ = T_1 \frac{\left(1 - e^{2\mu_s \beta}\right)}{\left(1 + e^{\mu_s \beta}\right)} = F_p \underbrace{\frac{\left(1 - e^{2\mu_s \beta}\right)}{\left(1 + e^{\mu_s \beta}\right)^2}}_{\mu} \end{aligned} \tag{50}$$

Thus, for both drives, the connection between the clamping and friction force is given by $F_f = \mu F_p$.

3.7.4 Slip and power analysis

The input power on the driving gear is $P_1 = F_f v_1 = M_1 \omega_1$, the output power is $P_2 = F_f v_2 = M_2 \omega_2$. In the case of the friction and belt drive, there is a slip between the driving and the driven machine:

$$s = \frac{v_1 - v_2}{v_1} = 1 - \frac{v_2}{v_1}, \tag{51}$$

while for the gear drive, $s = 0$. The torque, revolution number and power of the *driving* machine are M_1, n_1 and P_1, respectively. On the driven side, we have

$$M_2 = F_f R_2 = F_f R_1 \frac{R_2}{R_1} = M_1 \frac{R_2}{R_1}, \tag{52}$$

$$n_2 = \frac{\omega_2}{2\pi} = \frac{v_2}{R_2 2\pi} = (1-s)\frac{v_1}{R_1 2\pi}\frac{R_1}{R_2} = n_1(1-s)\frac{R_1}{R_2} \quad \text{and} \tag{53}$$

$$P_2 = M_2\omega_2 = P_1(1-s), \tag{54}$$

with which the efficiency of the drive is

$$\eta = \frac{P_2}{P_1} = \frac{v_2}{v_1} = \frac{v_1 - (v_1 - v_2)}{v_2} = 1 - s. \tag{55}$$

3.8 Load factor, efficiency and losses of machines

The *nominal (useful) power* or *rated power* is the output power a machine was designed for. This number is fixed, usually given on the nameplate. For example, we often say that "the power of a car is 120 hp "or "a 60kW electric motor is built into the system". This is the nominal power of the machine, but it should be clear that this is not the only power the machine is capable of producing; e.g. when we sit in our car in a traffic jam with running motor, the output power is zero.

The *load factor* x is defined as the ratio of the actual useful power and the nominal power:

$$x = \frac{P_u}{P_n}. \tag{56}$$

Thus, if $P_u < P_n$, we speak about *under*load and the case when $P_u > P_n$ is called *over*load. If there is no useful work (e.g. the car is at rest but the motor runs or the computer is switched on but no user programs are running), we speak about *idle run*.

It is important to know how the efficiency of the machine depends on the load factor. The input work covers the useful work plus the losses:

$$P_{i(\text{nput})} = P_{u(\text{seful})} + P_{l(\text{oss})} \tag{57}$$

The loss consists of two parts; one being independent of the load (constant loss, P_c) and the other increasing with increasing load (variable loss, P_v):

$$P_l(x) = P_{c(\text{onstant})} + P_{v(\text{ariable})}(x) \tag{58}$$

An example of constant loss is friction of the rotating parts (e.g. bearings), which has to be covered even in idle run. An example of variable loss is the increasing heat generated as the load increases. We have

$$P_v = x^n P_{v1}, \tag{59}$$

where P_{v1} is some constant (the subscript '1' denotes that variable loss at $x = 1$) and

- $n \approx 1$ for mechanical machines (e.g. pulley)
- $n \approx 2$ for electronic machines (e.g. electric motor) and
- $n \approx 3$ for hydraulic machines (e.g. pump, fan).

The above relations allow us to give the efficiency as a function of the load factor:

$$\eta(x) = \frac{P_u}{P_i} = \frac{xP_n}{\underbrace{xP_n}_{\text{useful}} + \underbrace{P_c + x^n P_{v1}}_{\text{losses}}} \tag{60}$$

Let us find the load with maximum-efficiency:

$$0 = \frac{d\eta(x)}{dx} = \frac{d}{dx} \frac{\overbrace{xP_n}^{f(x)}}{\underbrace{xP_n + P_c + x^n P_{v1}}_{g(x)}} \left(= \frac{f'g - g'f}{g^2} \right)$$
$$= \frac{P_n\left(xP_n + P_c + x^n P_{v1}\right) - xP_n\left(P_n + nx^{n-1}P_{v1}\right)}{\left(xP_n + P_c + x^n P_{v1}\right)^2}$$
$$= \frac{P_n\left(P_c + (1-n)x^n P_{v1}\right)}{\left(xP_n + P_c + x^n P_{v1}\right)^2} \quad \rightarrow \quad P_c = (n-1)x^n P_{v1} \tag{61}$$

Thus, we find that

for mechanical machines (n=1) there is no best-efficiency point but efficiency increases as the load is increased and $\eta \to 100\%$ as $x \to \infty$.

for electric machines (n=2) we have $P_c = x_{opt}^2 P_{v1}$, i.e. *at the best-efficiency point the constant and variable losses equal.* In other words, $x_{opt} = \sqrt{P_c/P_{v1}}$.

for hydraulic machines (n=3) we have $P_c = 2x_{opt}^3 P_{v1}$, i.e. *at the best-efficiency point the constant loss is the double of the variable loss.* In other words, $x_{opt} = \sqrt[3]{P_c/(2P_{v1})}$.

See illustration for mechanical and electrical machines in Fig.27.

3.9 Average load and efficiency

Suppose that a machine runs at varying loads during some period of time, but with constant load in each intervals. For example, from 8am to 10am x=50%, from 10am to 1pm x=90%, from 1pm to 2pm x=0% (launchtime) and finally, from 2pm to 5pm 110%.

We wish to calculate the average load $\overline{x}$ and efficiency $\overline{\eta}$ during the day. Both of them will be defined with the help of work as follows:

$$\overline{\eta} = \frac{W_{\text{useful}}}{W_{\text{input}}} \quad \text{and} \quad \overline{x} = \frac{W_{\text{useful}}}{W_{\text{nominal}}} \tag{62}$$

If we have N periods and the length of the ith period is denoted by t_i and the (constant) load and efficiency is x_i and η_i, we have

$$\overline{x} = \frac{W_{\text{useful}}}{W_{\text{nominal}}} = \frac{\sum_{i=1}^N t_i P_{u,i}}{\sum_{i=1}^N t_i P_n} = \frac{\sum_{i=1}^N t_i x_i \cancel{P_n}}{\sum_{i=1}^N t_i \cancel{P_n}} = \frac{\sum_{i=1}^N t_i x_i}{\sum_{i=1}^N t_i} \tag{63}$$

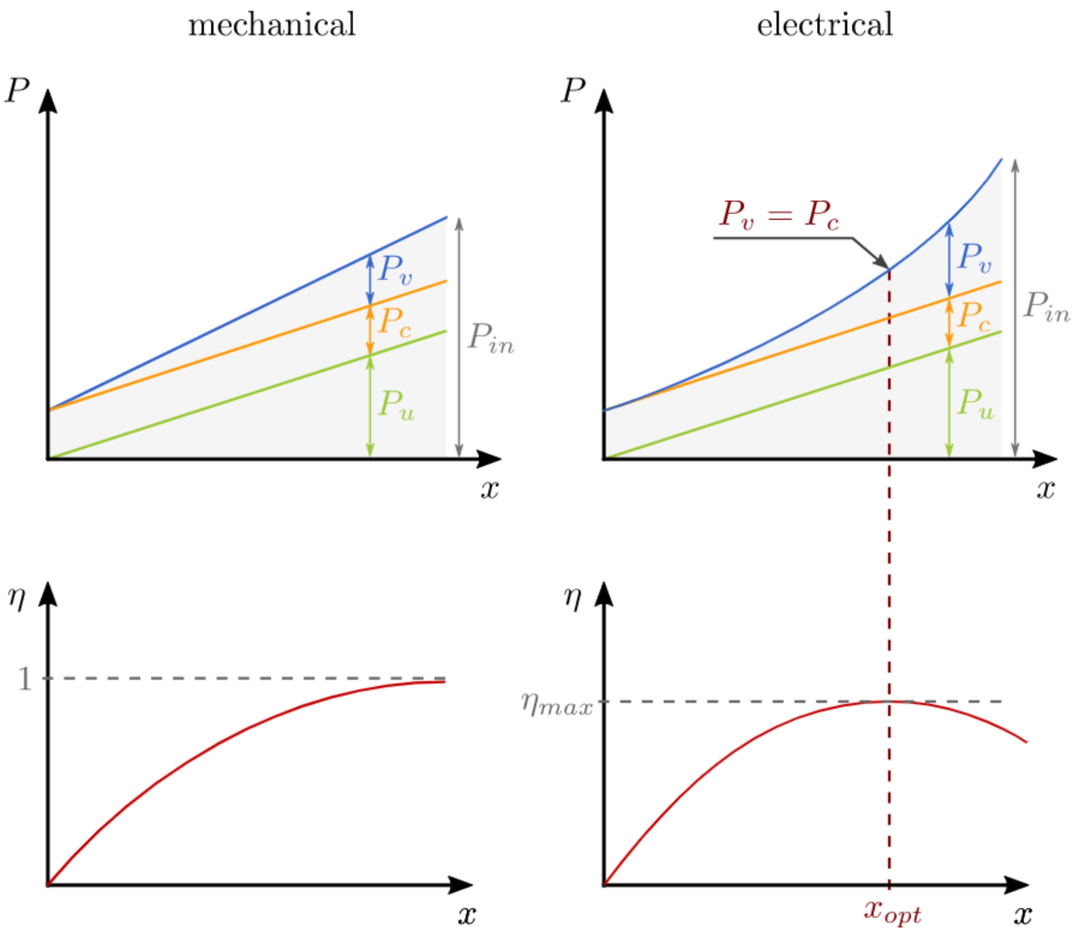


Figure 27: Power and efficiency as a function of the load factor for mechanical and electrical machines

and

$$\overline{\eta} = \frac{W_{\text{useful}}}{W_{\text{input}}} = \left.\frac{\sum_{i=1}^{N} t_i P_{u,i}}{\sum_{i=1}^{N} t_i P_{i,i}}\right|_{\eta_i \neq 0} = \frac{\sum_{i=1}^{N} t_i P_{u,i}}{\sum_{i=1}^{N} t_i \frac{P_{u,i}}{\eta_i}} = \frac{\sum_{i=1}^{N} t_i x_i}{\sum_{i=1}^{N} \frac{t_i x_i}{\eta_i}} \tag{64}$$

Note that the last expression of the above equation for $\overline{\eta}$ (including $t_i x_i/\eta_i$) is only valid if $\eta_i \neq 0$, i.e. there is no idle operation during the intervals!

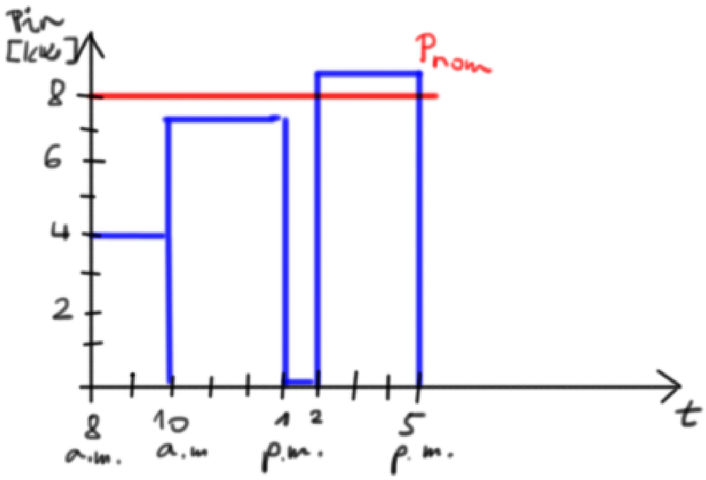


Figure 28: Varying load on a machine

3.10 Problems

Springs

Problem 3.1 A spring is initially compressed by 5 mm. We compress it further to 19 mm. The spring stiffness is 30 $\frac{N}{mm}$.

- Compute the energy increase of the spring during the action. $E_s = \frac{1}{2}k(y_1^2 - y_0^2) = 5.04$ J
- Calculate the sum of the elastic energy of the spring. $\Sigma E_s = \frac{1}{2}ky_1^2 = 5.415\ J$

Problem 3.2 A spring with stiffness $s = 100\,\mathrm{N/mm}$ is compressed from its initial length of $L_0 = 20\mathrm{cm}$ to $L_1 = 10\mathrm{cm}$.

- Find the force. We have $F = s\Delta l = 100\frac{\mathrm{N}}{\mathrm{mm}} \times 100\mathrm{mm} = 10^4\mathrm{N} = 10\mathrm{kN}$
- Find the work. We have to be carefull here: as we compress the spring more and more, the force increases:

$$F = \int_{x=0\mathrm{mm}}^{x=100\mathrm{mm}} F(x)dx = \int_{x=0\mathrm{mm}}^{100\mathrm{mm}} sx\,dx = s\left[\frac{x^2}{2}\right]$$
$$= 100\,\frac{\mathrm{N}}{\mathrm{mm}}\,\frac{(100\mathrm{mm})^2}{2} = 0.5 \times 10^6\mathrm{Nmm} = 0.5\,\mathrm{kJ}$$

Problem 3.3

- The stiffness of two springs are $k_1 = 51\ \frac{N}{mm}$ and $k_2 = 23\ \frac{N}{mm}$. Calculate the reduced spring stiffness of the springs connected in *series*. ($k_{red} = 15.85\ \frac{N}{mm}$)
- Calculate the total expansion of the springs if a force of 890 N is applied to the system. ($y = 56.15\ mm$)

Problem 3.4

- The stiffness of two springs are $k_1 = 34\ \frac{N}{mm}$ and $k_2 = 61\ \frac{N}{mm}$. Calculate the reduced spring stiffness of the springs connected in *parallel*. ($k_{red} = 95\ \frac{N}{mm}$)
- Compute the amount of force that we need to apply in order to get a compression of 30 mm. ($F = 2850\ N$)

Problem 3.5 Three spring are in series connection. Their spring constants are $k_1 = 4200\ \frac{\mathrm{N}}{m}$, $k_2 = 11300\ \frac{\mathrm{N}}{m}$ and $k_3 = 5200\ \frac{\mathrm{N}}{m}$.

(a) Find the reduced spring constant in $\frac{\mathrm{N}}{m}$. ($k_{red} = 1927\ \frac{\mathrm{N}}{m}$)

(b) Find the force of the spring in N, if the displacement is $x = 20$ mm. ($F = 38.54$ N)

(c) Find the energy stored in the springs in J. ($E = W = 0.3854$ J)

Force components, inclined plane

Problem 3.6 A timber with mass $m = 150$kg is pulled horizontally on a flat dirty road. The pulling chain is fixed to the centre of gravity of the timber. The coefficient of friction between the the wood and the ground is $\mu = 0.48$. Find the force needed to pull the timber! ($F = 706.3$N)

Problem 3.7 Find the tractive force needed to tow a car with mass $m = 2000$kg upwards on slope whose gradient is 4% if the rolling resistance is $C_{rr} = 0.028$ and the angle between the force and the plane of the slope is $\beta = 30°$. ($F = 1515$N) Calculate the tractive power if the velocity of the towing is $v = 50$km/h. (18.22kW)

Problem 3.8 There is a ramp whose gradient is adjustable and a block $m = 50$kg on it at rest. The slope of the ramp is slowly increased and continuously measured. It is found that at $\alpha = 7°$ the block begins to move.

- Calculate the the coefficient of static friction and the friction force! ($\mu_s = 0.12, F_f = 59.78$N)
- Express the general relation between the angle of the slope and the coefficient of static friction.
- Find the friction force when the angel of the slope is $\alpha = 3°$. ($F_f = 25.67$N)

Problem 3.9 A car with mass $m = 1200$kg, is *pushed* upwards on a slope with constant velocity. The gradient of the slope is 6% and the angle between the pushing force ($F = 1100$N) and the horizontal direction is $\gamma = 10°$. Calculate the rolling resistance, the work performed on a 500m long distance and the part of it, that is invested in to overtake the friction! ($C_{rr} = 0.03$, $W = 534.95$kJ, $W_f = 182.39$kJ)

Problem 3.10 A cylindrical object with mass $m = 80$ kg is pushed in a horizontal plane. The rolling fiction coefficient is $C_{rr} = 0.3$.

(a) Find the force normal to the ground in N. ($N = 784.8$ N)

(b) Find the maximum friction force in N. ($F_{\max} = 235.4$ N)

(c) Find the pushing force in N, if we want to object to accelerate with $a = 2\ \frac{\mathrm{m}}{\mathrm{s}^2}$. ($F_p = 395.4$ N)

Problem 3.11 An object is pulled with a rope in a horizontal plane. The angle between the ground and the rope is $\beta = 30°$. The mass of the body is $m = 35$ kg, and the friction coefficient is $\mu = 0.5$. Our aim is to determine the pulling force, if we want to accelerate the body with $a = 5\ \frac{\mathrm{m}}{\mathrm{s}^2}$!

(a) Write down the force balance equations in the x (horizontal) and y (vertical) directions. ($F = 310.6$ N)

(b) Find the necessary pulling force F. ($N = 188.0$ N)

(c) Find the friction force. ($F_f = 94.02$ N)

Friction drive

Problem 3.12 How much power can be transmitted with a friction drive, if the coefficient of friction between the wheels is $\mu = 0.37$? The diameter of the driven wheel is $D = 45$mm, its revolution number is $n = 120$rpm, and the clamping force is $F = 238$N. There is no slip between the wheels. ($P = 24.9$W)

Problem 3.13 $P = 0.4$kW power is to be transmitted with a friction drive. The wheels are supposed to roll without slip. The coefficient of friction between the wheels is $\mu = 0.4$. The diameter of the driven wheel is $D_2 = 16$cm and the applied gear ratio is $i = 2.1$ (i.e. reducing translation).

- Find the diameter of the driving wheel! ($D_1 = 7.62$cm)
- Find the clamping force (the force pushing the two fears against each other) if the revolution number of the driving shaft is $n = 2010$rpm! ($F = 124.7$N)

Problem 3.14 The diameter of the driving wheel of the friction drive is $D_1 = 20$ mm, and the diameter of the driven wheel is $D_2 = 35$ mm. Speed of the driving wheel is $n = 100\ \frac{1}{\mathrm{min}}$, and its torque is $M = 12.3$ Nm. The slip of the friction drive is $s = 5\%$.

(a) Find the torque of the driven wheel. ($M_2 = 21.53$ Nm)

(b) Find the speed of the driven wheel. ($n_2 = 0.9048\ \frac{1}{\mathrm{s}}$)

(c) Find the angular frequencies of both wheels. ($\omega_2 = 5.685\ \frac{\mathrm{rad}}{\mathrm{s}}$)

(d) Find the power of both wheels. ($P_1 = 128.8$ W, $P_2 = 122.4$ W)

(e) Find the efficiency of the drive. ($\eta = 0.95$)

Belt drive

Problem 3.15 An electric motor of 14kW and 1460rpm drives a machine via a flat belt drive.

- Find the diameter of the driving pulley, if the maximal allowed belt velocity is 30m/s.
- Find the diameter of the driven pulley, if the desired transmission is 2 (i.e. decelerating transmission) and we are expecting 3% slip.
- Find the torque of the driven pulley.

Pulley

Problem 3.16 An electric motor drives a pulley system consisting of two standing and two moving pulleys. The system lifts a mass of 2t, the lift speed is 10 cm/s. The losses of the system can be neglected.

- Find the rope force (4.91N) and the power need of the lift ($P = 1.96$W).
- Assuming 10cm pulley radius, find the required motor torque (491Nm) and revolution number (38.2 rpm).

Load factor, losses

Problem 3.17 The efficiency of an electric generator as a function of the output power has been measured. At full load the useful power was $P_{u,x=1} = 380$kW while the efficiency was $\eta = 95\%$. The same efficiency was measured when the output power was 200kW. Find the constant and variable loss at full load. ($P_c = 6.9$kW, $P_{v,x=1} = 13.1$kW)

Problem 3.18 The useful power of an elevator is $P_u = 4$kW, the efficiency at this point is $\eta = 66\%$. The nominal useful power of the elevator is $P_N = 6.4$kW, while the constant loss is $P_c = 0.8$kW. The variable loss is linearly proportional to the load, i.e. the machine is mechanical. Find the variable loss at full load. ($P_{v,x=1} = 2.02$kW) Calculate the efficiency at full, half, and quarter loads. ($\eta_{x=1} = 69.4\%$, $\eta_{x=0.5} = 63.9\%$, $\eta_{x=0.25} = 55\%$)

Problem 3.19 $P_u = 140$kW useful power is given by a transformer at full load, while its input power is $P_i = 148$kW. The transformer reaches its optimal efficiency at $x = 60\%$ load. Find the constant loss of the transformer. ($P_c = 2.12$kW) Find the optimal efficiency. ($\eta_{opt} = 95.2\%$)

Problem 3.20 Consider the pulley system displayed in figure 29. The mass of the lifted object is $m = 10$ kg.

(a) Find the force F needed to pull the object with a constant speed in N. ($F = 32.70$ N)

(b) Find the work needed to pull up to object to a height of $h = 4$ m! The answer is expected in J. ($E = W = 392.4$ J)

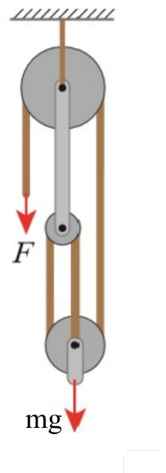


Figure 29: Schematics of the problem

(c) Find the force F in N, if the object is accelerating with $a = 1\ \frac{\mathrm{m}}{\mathrm{s}}$. ($F = 36.03$ N)

Problem 3.21 An object with a mass of $m = 9$ kg us pulled with a pulley. The diameter of the pulley is $D = 12$ cm, the radius of the shaft is $r = 5$ mm, and the friction coefficient of the shaft is $\mu = 0.01$. Find the traction force F_t needed to move the object with a constant speed, if the traction force is oriented downwards and angle between the traction force and a vertical line is $\alpha = 60°$. ($F_t = 88.42$ N)

(*Hint: you can use the law of cosines to calculate the normal force N!*)

Average power and efficiency

Problem 3.22 A machine repeats a *complete* working cycle periodically that lasts for $t = 40$min. During the working period the input power is $P_{i,w} = 8$kW and the efficiency is $\eta_w = 79\%$. Between two working periods, while the work piece is replaced, the machine runs idle with a power consumption of $P_{i,i} = 1.3$kW.

- Find the time available for the replacement of the work piece if a minimum of $\eta_{avg} = 75\%$ average efficiency must be kept. ($t = 9.9$min)
- Find the average load factor if the time of the replacement reduces to $t = 8$min. ($x_{avg} = 80\%$)

Problem 3.23 A prime mover works daily for $t_1 = 4$ h at full load with an efficiency of $\eta_1 = 78\%$, $t_2 = 3$ h with a load factor of $x_2 = 0.8$ and an efficiency of $\eta_2 = 76\%$, and $t_3 = 1$ h with a load factor of $x_3 = 0.3$ and an efficiency of $\eta_3 = 56\%$. Find the average load factor and average efficiency. ($x_{avg} = 0.838$, $\eta_{avg} = 76\%$)

Problem 3.24 The input power of an *electronic* machine is $P_{in} = 23$ kW, the nominal power is $P_n = 20$ kW, and the load factor at the current operating point is $x = 90\%$. The constant loss (which are independent of the load factor) is $P_c = 1.3$ kW

(a) Find the useful power. ($P_u = 18$ kW)

(b) Find the power loss. ($P_l = 5$ kW)

(c) Find the coefficient of the variable power formula! ($P_{v1} = 4.568$ kW)

(d) Find the optimal load factor (the load factor of the highest efficiency point). ($x_{opt} = 0.5335$)

(e) Find the maximum efficiency. ($\eta_{max} = 80.41\%$)

Problem 3.25 The nominal power of an internal combustion engine is $P_n = 20$ kW. First, the useful power of the machine is $P_1 = 12$ kW for $t_1 = 30$ min, then it is changed to $P_2 = 9$ kW for $t_2 = 45$ min, and at the end of the operation its value is $P_3 = 18.5$ kW for $t_3 = 85$ min. The input powers of the machine during the operation are $P_{in,1} = 23$ kW, $P_{in,2} = 26.5$ kW, and $P_{in,3} = 30.3$ kW, respectively.

(a) Find the load factor for each interval (x_1, x_2, x_3)!

(b) Find the efficiency for each interval (η_1, η_2, η_3)!

(c) Find the power loss for each interval! $(P_{l,1}, P_{l,2}, P_{l,3})$!

(d) Find the useful work for each interval! (W_1, W_2, W_3)!

(e) Find the loss work for each interval! $(W_{l,1}, W_{l,2}, W_{l,3})$!

(f) Find the mean load factor $(\overline{x})$!

(g) Find the mean efficiency! $(\overline{\eta})$!

Solution (the given data is highlighted with the color yellow)

Interval	1	2	3	4	sum/mean
t_j $(-)$	0.500	0.750	0.250	1.417	2.917
P_j $(-)$	12.00	9.00	0.00	18.50	13.36
$P_{in,j}$ $(-)$	23.00	26.50	15.00	30.30	26.76
x_i $(-)$	0.6000	0.4500	0.0000	0.9250	0.6679
η_i (%)	52.17	33.96	0.00	61.06	49.91
Pl, i (kW)	11.00	17.50	15.00	11.80	13.4029
W_i (kWh)	6.00	6.75	0.00	26.21	38.9583
$W_{l,i}$ (kWh)	5.50	13.13	3.75	16.72	39.0917

4 Fluid mechanics

4.1 Introduction

In many engineering systems we come across machines, which either do work on fluids (pumps, compressors, ventilators) or extract energy from fluids (water turbines or wind turbines). Examples of fluids include gases and liquids. Typically, liquids are considered to be incompressible, whereas gases are considered to be compressible. However, there are exceptions in everyday engineering applications (we shall return to this issue later). By definition, a fluid is a material continuum that is unable to withstand a static shear stress. Unlike an elastic solid which responds to a shear stress with a recoverable deformation, a fluid responds with an irrecoverable flow. The following physical quantities are of great importance:

Pressure (symbol: p, SI unit: $Pa = N/m^2 = kg/(ms^2)$) is the force per unit area applied in a direction perpendicular to the surface of an object: $p = F/A$. If the term 'pressure' is used, it means absolute pressure, whose zero point is the (full) vacuum. Often we deal with relative (or gauge) pressure p_r, which is the pressure relative to the ambient pressure $p_0 \approx 10^5$Pa (the exact value varies when and where the measurement was taken):

$$p_r = p - p_0. \tag{65}$$

Note that relative pressure can be negative, meaning vacuum. A third way or representing (vacuum) pressure is relative vacuum:

$$rel.vac. = \frac{p_0 - p}{p_0} \times 100\% \tag{66}$$

Thus 0% relative vacuum means ambient pressure p_0 and 100% relative vacuum means that $p = 0$ ($p_r = -10^5$)Pa.

Density (symbol: ρ, SI unit: kg/m^3) is defined as mass per unit volume, $\rho = m/V$. The density of a substance is the reciprocal of its specific volume ν, a representation commonly used in thermodynamics.

Compressibility is a measure of the relative volume change of a fluid as a response to a pressure change.

Fluids can be classified into ideal and real fluids:

Ideal fluids are homogeneous, incompressible and no internal friction occurs. In many practical cases, liquids can be considered to be ideal fluids.

Real fluids are compressible. For example, for air, we know from the *ideal gas law* that $\rho = p/(RT)$, thus if air is compressed from 1 bar to 2 bar pressure by an isothermal process (i.e. temperature remains constant), its density will double. For liquids, we have $\Delta p/B = \Delta V/V$, where Δp is the pressure change, V is the initial volume, ΔV is the volume change and B is the bulk modulus. For example, for water $B_w = 2.1 \times 10^9 \mathrm{Pa} = 2.1\,\mathrm{GPa}$ and for steel $B_s = 160\,\mathrm{GPa}$. Friction is also present in real fluids, which manifests itself in *pressure loss*. This simply means that e.g. the pressure along a horizontal pipe decreases in the direction of flow. This is an important issue that must be taken into account when dealing with fluid flow systems.

4.2 Hydrostatic pressure and Pascal's law

Pascal's law is a principle in fluid mechanics given by Blaise Pascal that states that a pressure change at any point in a confined incompressible fluid is transmitted throughout the fluid such that the same change occurs everywhere. Alternatively, one can say that the pressure applied to any part of the enclosed liquid will be transmitted equally in all directions through the liquid. As an example, consider the hydraulic jack depicted in Fig. 30. As the pressure is the same everywhere, we have

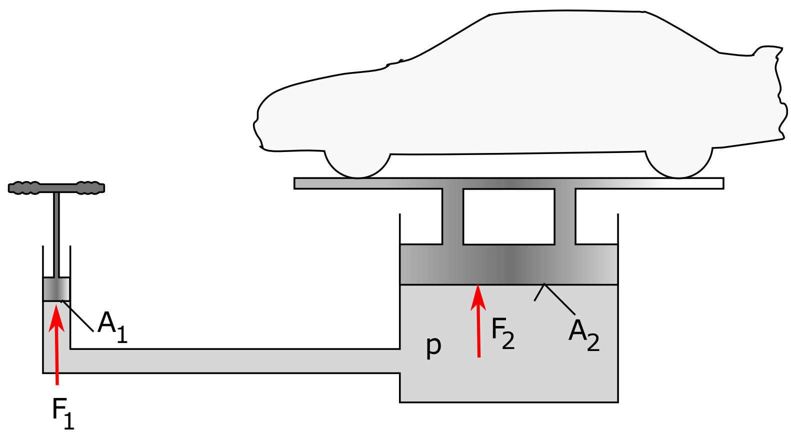


Figure 30: Hydraulic jack

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \rightarrow \quad \frac{F_1}{F_2} = \frac{A_1}{A_2}, \tag{67}$$

which is essentially a simple machine, changing the force, just like a gear drive changes the torque.

Hydrostatic pressure is the pressure that is exerted by a fluid at equilibrium at a given point within the fluid, due to the force of gravity. Hydrostatic pressure increases in proportion to depth measured from the surface because of the increasing weight of fluid exerting downward force from above. This means that

$$\Delta p = \rho g \Delta h, \tag{68}$$

where Δp is the hydrostatic pressure (given in pascals in the SI system), ρ is the fluid density (in kilograms per cubic meter in the SI system), g is acceleration due to gravity and Δh is the height of the fluid column. It is important to mention, that the hydrostatic pressure does not depend on the shape of the vessel, see Fig. 31.

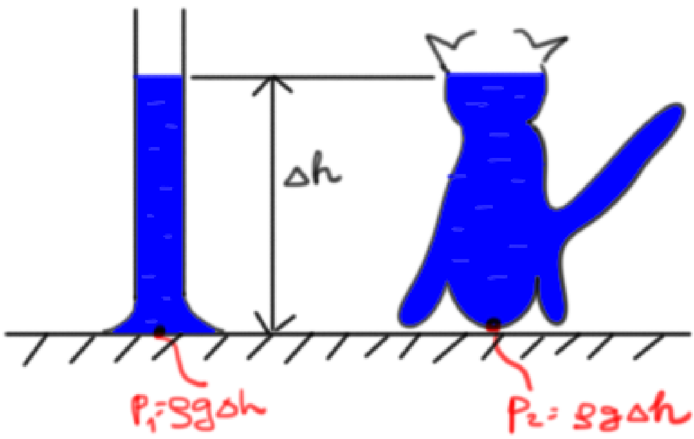


Figure 31: Hydrostatic pressure does not depend on the shape of the vessel

4.3 Mass conservation - law of continuity

In fluid mechanics, mass conservation means that, under steady-state conditions, the amount of material entering and leaving a system per unit time equals. We define *mass flow rate* as the mass of substance which passes through a given surface per unit time and it can be calculated from the density of the substance, the cross sectional area through which the substance is flowing, and its velocity relative to the area of interest:

$$\dot{m} = \rho A v_\perp, \tag{69}$$

where $\dot{m}$ is the mass flow rate (kg/s), ρ is the density, $v_\perp$ is the velocity component *perpendicular to* A and A is the flow-through area. This is equivalent to multiplying the *volumetric flow rate* Q by the density:

$$\dot{m} = \rho Q, \tag{70}$$

where Q is the volumetric flow rate, its SI unit is m^3/s.

Now revisit the hydraulic jack problem (Fig. 30). The mass conservation says that

$$\rho A_1 v_1 = \rho A_2 v_2 \tag{71}$$

and from this one can obtain

$$v_2 = v_1 \frac{A_2}{A_1} \tag{72}$$

4.4 Energy conservation - Bernoulli's equation

Bernoulli's principle can be derived from the principle of conservation of energy. This states that in a steady flow the sum of all forms of mechanical energy in a fluid along a streamline is the same at all points *on that streamline*:

$$\underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}} + \underbrace{mgh}_{\text{potential energy}} + \underbrace{pV}_{\text{work done by pressure}} = \text{constant.} \tag{73}$$

However, the above equation is not useful in fluid mechanics, as it is not clear what is meant by the velocity of the fluid of mass m - the velocity changes in any small volume of fluid. Thus, we divide the above equation by V to obtain the energy content of an arbitrarily small volume:

$$\frac{\rho}{2}v^2 + \rho g h + p = \text{constant.} \tag{74}$$

Let us consider two points on a streamline (the path of a fluid particle), denoted by 1 and 2, the flow is from 1 to 2. Than, if the fluid is ideal (no friction and incompressible) and the flow is steady, we have

$$p_1 + \frac{\rho}{2}v_1^2 + \rho g h_1 = p_2 + \frac{\rho}{2}v_2^2 + \rho g h_2. \tag{75}$$

4.5 Application 1 - flow in a confuser

Consider the flow in a pipe of decreasing diameter (confuser). The diameter of the inlet is D_1, the pressure and velocity of the fluid is p_1 and v_1. The diameter of the outlet is $D_2 < D_1$, the pressure and velocity of the fluid here is p_2 and v_2.

By virtue of the continuity equation, we have

$$\rho v_1 A_1 = \rho v_2 A_2 = \rho Q, \tag{76}$$

which implies that as $A_2 < A_1$, $v_2 > v_1$. Substituting the cross section at both sides we obtain

$$\rho v_1 \frac{D_1^2 \pi}{4} = \rho v_2 \frac{D_2^2 \pi}{4} \tag{77}$$

$$v_2 = v_1 \left(\frac{D_1}{D_2} \right)^2 \tag{78}$$

If we introduce a general coordinate x starting with zero at the beginning of the confuser we got the formula

$$v(x) = v_1 \left(\frac{D_1}{D(x)} \right)^2 \tag{79}$$

Applying Bernoulli's equation between points 1 and 2, we obtain

$$p_1 + \frac{\rho}{2} v_1^2 + \cancel{\rho g h_1} = p_2 + \frac{\rho}{2} v_2^2 + \cancel{\rho g h_2}, \tag{80}$$

from which wee see that if $v_2 > v_1$, we have $p_2 < p_1$. Expressing p_2 and substituting the results of (78) results in

$$p_2 = p_1 + \frac{\rho}{2}(v_1^2 - v_2^2) = p_1 + \frac{\rho}{2} v_1^2 \left(1 - \left(\frac{D_1}{D_2} \right)^4 \right) \tag{81}$$

If we use the same general coordinate x defined above we obtain

$$p(x) = p_1 + \frac{\rho}{2} v_1^2 \left(1 - \left(\frac{D_1}{D_(x)} \right)^4 \right) \tag{82}$$

See the diagram plotted by (79) and (82) in Fig. 32.

4.6 Application 2 - pressure measurement with U-tube

An easy example of a measurement with U-tube is given here while You find a most complex case in the Problems section below. The U-tube is a U-shaped tube typically made of glass and equipped with a scale in order to make reading the length of fluid coloumn easy. One or two side of the tube is/are connected to the point where we are interested in the pressure or between two points where the pressure difference is the question. The tube is filled with a different, non-solvable fluid compared to the fluid whose pressure has to be determined. This can be e.g. mercury if we determine the pressure of water.

The manometer equilibrium of the hydrostatic pressures in the left and right side for the setup shown in Fig.33 is:

$$p_0 + \varrho_{mercury} g h_2 = p_1 + \varrho_{water} g h_1 \tag{83}$$

from which p_1 can be easily obtained.

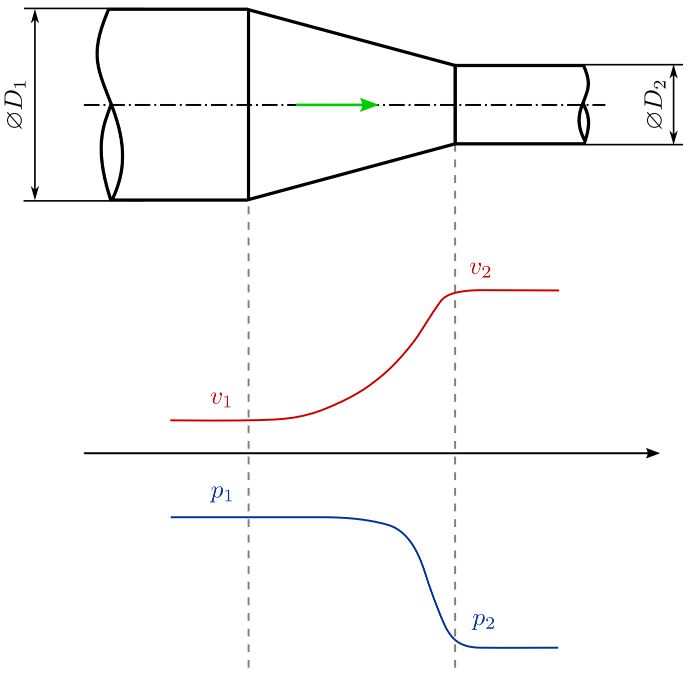


Figure 32: Flow in a confuser.

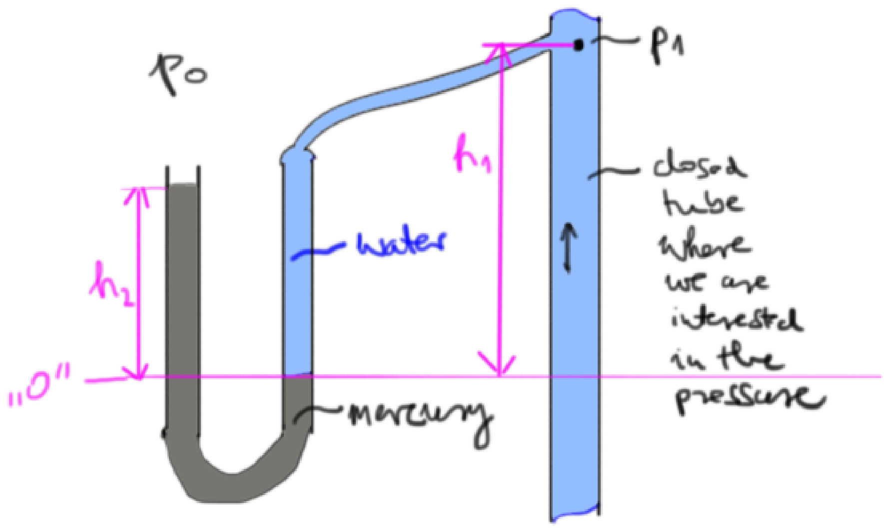


Figure 33: Measurement with u-tube manometer

4.7 Problems

Problem 4.1 There is a given absolute pressure of $p = 420\text{mbar}$. Calculate the gauge pressure in pascals and the relative vacuum in percentage!

($p_g = -58\text{kPa}$, $rel.vac = 58\%$)

Problem 4.2 The plunger of a hypodermic syringe is pressed with a constant velocity of $v_1 = 0.9\text{cm/s}$. The diameter of the plunger is $D = 1.2\text{cm}$, while the radius of the hollow in the needle is $r = 0.4\text{mm}$ (see Fig.34). Find the velocity of the fluid exits at the end of the needle! ($v_2 = 2.025\text{m/s}$)

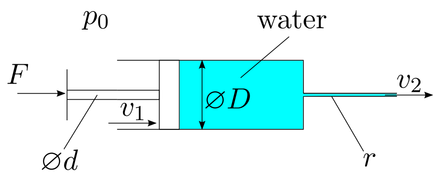


Figure 34: Sketch of Problem 4.2

Problem 4.3 Water flows through a tapering pipe with a volumetric flow rate of $Q = 115\text{dm}^3/\text{min}$. The radius at the beginning of the pipe is $r_1 = 62\text{mm}$, while the diameter at the end is $D_2 = 46.2\text{mm}$. Calculate the velocities at both ends! ($v_1 = 0.159\text{m/s}$, $v_2 = 1.143\text{m/s}$)

Problem 4.4 Air with a temperature of $T_1 = 90°\text{C}$ and a velocity of $v_1 = 15\text{m/s}$ enters into a cylindrical tube of a heat exchanger. The diameter of the tube is $d = 130\text{mm}$. At the outlet of the heat exchanger the air is cooled down to $T_2 = 20°\text{C}$. The pressure can be assumed constant along the whole tube. According to measurements the density of the air is given in the form: $\rho[\text{kg/m}^3] = 352.977/T[\text{K}]$. Calculate the velocity and the mass flow rate at the end of the tube! ($v_2 = 12.1\text{m/s}$, $\dot{m} = 0.194\text{kg/s}$)

Problem 4.5 A compressor compresses air from $p_0 = 101.325\text{kPa}$ to $p_1 = 25\text{bar}$. The mass flow rate is $\dot{m} = 30\text{kg/s}$ and the change of state is isotherm. Calculate the density at the pressure side if it is $\rho_0 = 1.2041\text{kg/m}^3$ at the suction side and is given by the function $\rho(p) = \rho_0(p/p_0)^{0.714}$! ($\rho_1 = 11.877\text{kg/m}^3$) Find the velocities and volume flow rates at the suction and pressure side if the diameters of the connected pipes are $D_0 = 900\text{mm}$ and $D_1 = 400\text{mm}$ respectively! ($v_0 = 39.16\text{m/s}$, $v_1 = 20.1\text{m/s}$, $Q_0 = 24.91\text{m}^3/\text{s}$, and $Q_1 = 2.53\text{m}^3/\text{s}$)

Problem 4.6 There is mercury with a density of $\rho_m = 13600\text{kg/m}^3$ in a U-tube manometer whose both sides are open. How high water column ($\rho_w = 1000\text{kg/m}^3$) is needed to reach a $h_m = 16\text{mm}$ difference between the levels of mercury in the two sides? ($h_w = 218\text{mm}$)

Problem 4.7 There is a U-shaped glass tube filled partially with mercury ($\rho_m = 13600$ kg/m^3). One side of the tube is opened while the other is closed. In the closed side the level of the mercury is $\Delta h = 153\text{mm}$ higher than the level in the opened side. Calculate the pressure in the closed side above the mercury if the atmospheric pressure is $p_0 = 99\text{kPa}$! ($p = 78.6\text{kPa}$)

Problem 4.8 The barometer invented by Torricelli consists of a tube circa 1 meter long, sealed at the top end, filled with mercury, which is set vertically into a basin of mercury (see Fig.35). The pressure above the mercury in the tube can be neglected. Calculate the height of the mercury column ($\rho_m = 13600\text{kg/m}^3$) in the tube of the barometer on a $z = 2000$m high hill if the atmospheric pressure at the sea level is $p_0 = 101.325$kPa and the isothermal atmosphere model can be used for the pressure (i.e. $p(z) = p_0 e^{-1.166 \cdot 10^{-4} z}$ at $T = 20°$C)! ($h_m = 601$mm)

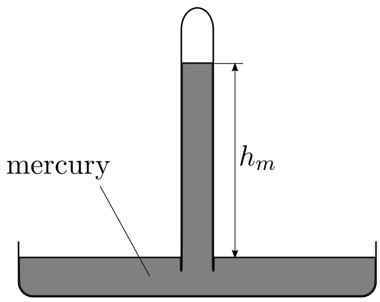


Figure 35: Sketch of Problem 4.8

Problem 4.9 Water ($\rho_w = 1000$ kg/m^3) flows through a pipe with a diameter of $d = 50$ mm. The volume flow rate is $Q = 72$ dm^3/min and there is a height difference of $h = 3.5$ m between the two ends of the pipe. The fluid flows upwards. Calculate the velocity and the pressure at the higher end if the pressure at the other end is $p_1 = 4$bar! ($v_2 = v_1 = 0.61$ m/s, $p_2 =$ 365.7 kPa)

Problem 4.10 There is a tank filled with water ($\rho_w = 1000\text{kg/m}^3$). The top of the tank is opened to the air, while there is a small outlet at the bottom (See Fig.36). The cross sectional area of the outlet is negligible comparing to the open surface of the water. Find the velocity of the water at the outlet if the water level is $h = 2$m high! Express the velocity as a function of the height of the water level! ($v_2 = 6.26$m/s, $v_2(h) = \sqrt{2gh}$)

Problem 4.11 Water ($\rho_w = 1000\text{kg/m}^3$) flows horizontally with a velocity of $v = 5$m/s. An L-shaped tube is placed into the flow as can be seen in Fig.37. Determine difference between the level in the tube and the surface of the flow! ($h = 1.27$m/s) Calculate the gauge pressure inside the tube at the level where 'z and h meets' and at the stagnation point if $z = 0.5$m! ($p_{g,l} = 4.9$kPa, $p_{g,stag} = 17.4$kPa)

Problem 4.12 In problem 3.2 calculate the force needed to press the plunger if the diameter of the rod is $d = 5$mm and the atmospheric pressure is $p_0 = 1$bar! ($F = 2.2$N)

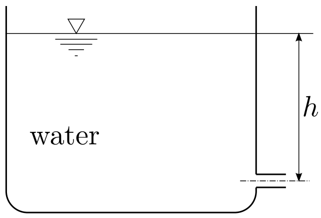


Figure 36: Sketch of Problem 4.10

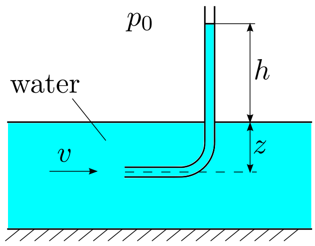


Figure 37: Sketch of Problem 4.11

Problem 4.13 Water ($\rho_w = 1000\text{kg/m}^3$) flows through a rising and tapering pipeline with a volume flow rate of $Q = 30\text{dm}^3/\text{s}$ (see Fig.38). The inclination of the pipeline is $\alpha = 20°$. The lower side of the pipe has a diameter of $D_1 = 200\text{mm}$, while the diameter of the upper side is $D_2 = 140\text{mm}$. A U-tube manometer filled with mercury ($\rho_m = 13600\text{kg/m}^3$) is connected to the pipeline. The length between the connection points is $l = 17\text{m}$. (The transferring fluid is also water.) Calculate the velocities and the difference between the levels of the mercury in the manometer! ($v_1 = 0.95\text{m/s}$, $v_2 = 1.95\text{m/s}$, $\Delta h = 473\text{mm}$)

Problem 4.14 The pressure inside a tank is $p_t = 10$ MPa, and the pressure inside a vacuum chamber is $p_c = 200$ hPa. The pressure of the ambient air is $p_0 = 1$ atm $= 101325$ Pa.

(a) Find the relative pressure in the tank, in both Pa and MPa. ($p_{r,t} = 9898675$ Pa $= 9.8987$ MPa)

(b) Find the relative pressure in the vacuum chamber, in both Pa and hPa. ($p_{r,c} = -81325$ Pa $= -813.25$ hPa)

(c) Find the relative vacuum in the vacuum chamber. ($p_{\text{rel},c} = 80.26\%$)

Problem 4.15 We use a bike pump to lift an object. The force applied to the bike pump is transferred through an ideal liquid in a closed space, so

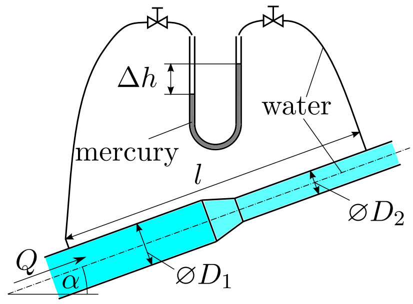


Figure 38: Sketch of Problem 4.13

Pascal's principle can be applied. The pressure of the fluid pushes the object on an area $A_2 = 0.2\ \mathrm{m}^2$, and the mass of the object is 30 t. The force we can push the bike pump with is $F_1 = 750$ N.

(a) Find the weight of the object. ($F_2 = 294.3$ kN)

(b) Find the area of the piston of the bike pump. ($A_1 = 5.0968 \cdot 10^{-4}\ \mathrm{m}^2 = 5.097\ \mathrm{cm}^2$)

Problem 4.16 A vertical tube is closed at its bottom, and its top is open to atmospheric pressure $p_0 = 1013$ hPa. The bottom of the tube is filled with mercury: the height of the mercury is $h_m = 23$ cm. On the mercury, there is a water column that's height is $h_w = 10$ cm. The densities of the mercury and the water are $\rho_m = 13600\ \frac{\mathrm{kg}}{\mathrm{m}^3}$ and $\rho_w = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$, respectively.

(a) Find hydrostatic pressure of the water! ($\Delta p_w = 981$ Pa)

(b) Find hydrostatic pressure of the mercury! ($\Delta p_m = 30690$ Pa)

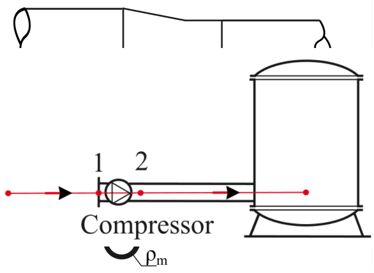
(c) Find total hydrostatic pressure of the two liquid columns! ($\Delta p_t = 4050$ Pa)

(d) Find total hydrostatic pressure at the bottom of the tube! ($\Delta p_b = 105350\ \mathrm{Pa} = 105.35$ kPa)

Problem 4.17

A U-tube manometer measures the pressure difference between two points in a pipe carrying water. The liquid in the U-tube which is used to measure the pressure difference is mercury, an on top of the mercury there is water (see figure). The displacement of the mercury column is $h = 17$ cm. The densities of the mercury and the water are $\rho_m = 13600\ \frac{\mathrm{kg}}{\mathrm{m}^3}$ and $\rho_w = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$, respectively. Find the pressure difference between the two points of the duct. ($\Delta p = 21013\ \mathrm{Pa} = 21.013$ kPa)

Problem 4.18



We convey air with a compressor. The mass flow rate of the air is $\dot{m} = 2.5\ \frac{\text{kg}}{\text{s}}$. The area of both the suction and pressure side is $A = 0.01227\ \text{m}^2$. The suction side is denoted with index (1), and the pressure side is denoted with (2). The pressures and temperatures at the pressure and suction side are the following: $t_1 = 30°C$, $t_2 = 221,85°C$, $p_1 = 1$ bar, $p_2 = 3$ bar. We can approximate the air as an ideal gas, therefore the ideal gas law holds: $\frac{p\ (\text{Pa})}{\rho\ \left(\frac{\text{kg}}{\text{m}^3}\right)} = R\ \left(\frac{\text{J}}{\text{kg}\cdot\text{K}}\right) \cdot T\ (\text{K})$. The specific gas constant is $R = 286\ \left(\frac{\text{J}}{\text{kg}\cdot\text{K}}\right)$.

(a) Find the density of the air at the suction side. ($\rho_1 = 1.153\ \frac{\text{kg}}{\text{m}^3}$)

(b) Find the volumetric flow rate at the suction side. ($q_1 = 2.168\ \frac{\text{m}^3}{\text{s}}$)

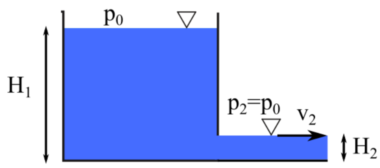
(c) Find the velocity of the air at the suction side. ($v_1 = 176.6\ \frac{\text{m}}{\text{s}}$)

(d) Find the density of the air at the pressure side. ($\rho_2 = 2.119\ \frac{\text{kg}}{\text{m}^3}$)

(e) Find the volumetric flow rate at the pressure side. ($q_2 = 1.180\ \frac{\text{m}^3}{\text{s}}$)

(f) Find the velocity of the air at the pressure side. ($v_2 = 96.13\ \frac{\text{m}}{\text{s}}$)

Problem 4.19



The height of the water level in a dam is $H_1 = 15$ m. The water can leave the dam at the bottom of the wall, at an orifice with a height of $H_2 = 1$ m. The density of the water is $\rho_w = 1000\ \frac{\text{kg}}{\text{m}^3}$, the atmospheric pressure is $p_0 = 1$ bar, and the dam is $B = 30$ m wide in the spanwise direction.

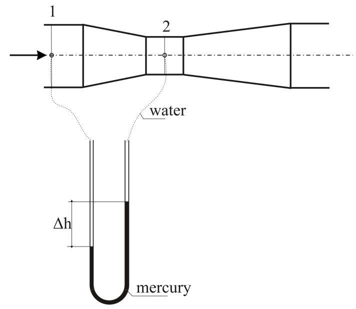
(a) Find the flow-through area $A = 30\ \text{m}^2$)!

(b) Using Bernoulli's equation, find the velocity of the water leaving the dam ($v_2 = 16.57\ \frac{\text{m}}{\text{s}}$)

(c) Find the volumetric flow rate of the water. ($q_2 = 497.2\ \frac{\text{m}^3}{\text{s}}$)

(d) Find the mass flow rate of the water. ($\dot{m}_2 = 497204\ \frac{\text{kg}}{\text{s}} = 497.204\ \frac{\text{t}}{\text{s}}$)

Problem 4.20 A Venturi tube is displayed in the figure. The cross section with the larger diameter is denoted with index (1), and the cross-section with the small diameter is denoted with index (2). The diameters of the tubes are $d_1 = 50$ mm and $d_2 = 10$ mm. Th fluid in the venturi tube is water, and it's pressure difference is measured with a U-tube, in which the liquid is mercury. The displacement of the mercury columns us $\Delta h = 13$ mm



(a) Find the flow-through area of the tube at point (1). ($A_1 = 0.002$ m^2)

(b) Find the flow-through area of the tube at point (2). ($A_2 = 7.854 \cdot 10^{-5}$ m^2)

(c) Find the pressure difference between the two points. ($\Delta p = 1607$ Pa)

(d) Find the velocity of the water at point (1). ($v_1 = 0.07177\ \frac{\mathrm{m}}{\mathrm{s}}$)

(e) Find the velocity of the water at point (2). ($v_2 = 1.794\ \frac{\mathrm{m}}{\mathrm{s}}$)

(f) Find the volumetric flow rate. ($q = 1.409 \cdot 10^{-4} \frac{\mathrm{m}^3}{\mathrm{s}}$)

(g) Find the mass flow rate. ($m = 0.1409\ \frac{\mathrm{kg}}{\mathrm{s}}$)

Problem 4.21 We convey water through a horizontal pipe. The length of the pipe is $L = 14$ m, and it's diameter is $d_1 = 50$ mm. The pipe friction coefficient is $\lambda = 0.01$, and the volumetric flow rate of the water is $q = 1.2 \cdot 10^{-2}\ \frac{\mathrm{m}^3}{\mathrm{s}}$.

(a) Find the flow-through area (A_1 (m^2))!

(b) Find the velocity of the water in the pipe (v_1 $\left(\frac{\mathrm{m}}{\mathrm{s}}\right)$)!

(c) Find the loss coefficient of the pipe segment! (ζ_1 $(-)$)!

(d) Find the pressure difference between the ends of the pipe segment! (Δp_1 (Pa))!

(e) We change the diameter of the pipe to $d_2 = 100$ mm! Repeat the calculation, and compare the pressure losses!

Solution:

	A (m^2)	v $\left(\frac{\mathrm{m}}{\mathrm{s}}\right)$	ζ $(-)$	Δp_1 (Pa)
$d = d_1$	0.001963	6.1115	2.8	52291
$d = d_2$	0.007854	1.5279	1.4	1634.1

5 Simple thermodynamic machines

5.1 Introduction

5.1.1 Enthalpy

Consider the problem of compressing a gas from pressure p_1 to p_2 inside a piston in two ways

- first, in a cylinder that is perfectly (heat) isolated and
- secondly, within a cylinder that has very thin (possibly metal) walls, with no heat isolation.

In the first case, not only we increase the pressure from p_1 to p_2 but also the temperature of the gas will increase from T_1 to T_2. In the second case, as the extra heat due to the compression will be conducted to the surrounings, at the end of the compression process, even though the pressure will be p_2, the gas temperature will remain T_1.

This simple example shows us that the energy content included in the Bernoulli equation $p + \frac{\rho}{2}v^2 + \rho gh$ is insufficient to capture the behaviour of gases. In thermodynamics, we often use the concept of the *ideal gas* (for teaching purposes) as an approximation for real gases. The ideal gas is a gas of particles considered as point objects that interact only by elastic collisions and fill a volume such that their mean free path between collisions is much larger than their diameter. Such systems approximate the monatomic gases, helium and the other noble gases. Here the kinetic energy consists only of the translational energy of the individual atoms. Monatomic particles do not rotate or vibrate, and are not electronically excited to higher energies except at very high temperatures.

The *internal energy* e of an ideal (chalorically perfect) gas is $e = c_v T$, where c_v is the specific heat constant measured at constant volume. Hence, the overall (total) enthalpy (that is, energy of unit mass, J/kg) of a gas is

$$h_t(J/kg) = \underbrace{\frac{1}{2}v^2}_{\text{kinetic energy}} + \underbrace{gh}_{\text{potential en.}} + \underbrace{\frac{p}{\rho}}_{\text{work done by pressure}} + \underbrace{c_v T}_{\text{internal en.}} \tag{84}$$

Often, one can neglect the changes in the kinetic and potenial energy; an example would be the case of compressing the gas in a piston: in such cases, nor the kinetic energy (the fluid is at rest before and after the compression) nor the potential energy changes. Thus, for ideal gases, the *thermodynamic enthalpy* is:

$$h = \frac{p}{\rho} + c_v T \bigg|_{p/\rho = RT} = (R + c_v)\, T = c_p\, T, \tag{85}$$

where c_p is the specific heat capacity measured at constant pressure and we made use of the ideal gas assumption $p/\rho = RT$.

In the case of liquids, the situation is more complicated. In general, two state variables (for example, $p-T$, $p-v$, $p-h$, etc. ($v = 1/\rho$ (m^3/kg) is the specific volume)) determine the rest of the state variables. Furthermore, c_v and c_p are also functions of the state variables. Generally, the relationship of the state variables available only via measurements, and the data is available in tables, and specific values can be calculated by using interpolation. In the case of water, e.g., such tables can be accessed via programs, available both on PC and mobile phones. However, there exist cases in which simpler formulae are also provide a sufficiently good approximation. For example, when the height difference and velocity change are negligible in the case of water (e.g., when pumping water in water distribution networks, or heating water by a few $10°C$), it follows from equation (84) that the enthalpy change is only due to the temperature and pressure change change. Furthermore, if the pressure change and temperature change are not extremely large, $c_v =$ const. and $c_p =$ const. are reasonable assumptions, therefore $c_p = c_v = c$. Therefore, the enthalpy change can be calculated as follows:

$$\Delta h = \frac{p}{\rho} + c\Delta T. \tag{86}$$

As an indication, some numerical values of the heat capacities are listed below. For air at 18^oC and $p = 1$ bar (absolute) pressure, we have $c_p =$ 1006 J/kg/K, $c_v = 717.1$ J/kg/K, and both of these values vary both with pressure and temperature! For water at room temperature, we have $c = 4200$ J/kg/K (the actual values are $c_v = 4164$ kJ/kg/K and $c_p = 4186$ kJ/kg/K, but since the difference is small because the volume change during heating is negligible we do not differentiate between c_p and c_v in such cases).

The power need of changing the enthalpy by $\Delta h = h_2 - h_1$ is

$$P = \dot{m}\Delta h \quad (kg/s \cdot J/kg = J/s = W). \tag{87}$$

5.1.2 Heating value, specific fuel consumption

The *heating value* (or energy value or calorific value) of a substance, usually a fuel (or food) is the amount of heat released during the combustion of a specified amount of it. The heating value is typically provided in MJ/kg as in table 6.

The heat production of burning $\dot{m}$ (kg/s) fuel per seconds with heating value H (MJ/kg) results in $Q = \mathrm{H} \cdot \dot{m}$ (MJ/s=MW) power.

Fuel	Hydrogen	Kerosene	Wood	Coal
H (MJ/kg)	142	46	22	15...32

Table 6: Heating values of a few common fuels

The *specific fuel consumption* of SFC is "how much fuel the engine burns to produce 1 J useful work", that is

$$\mathrm{SFC} = \frac{\text{mass of fuel burnt}}{\text{useful (shaft) work}} = \frac{m_{fuel}}{W_{useful}} = \frac{\dot{m}_{fuel}}{P_{useful}} \quad (kg/J). \tag{88}$$

5.2 Internal combustion engines

The internal combustion engine (ICE) is an engine in which the combustion of a fuel (normally a fossil fuel) occurs with an oxidizer (usually air) in a combustion chamber. In an internal combustion engine the expansion of the high-temperature and -pressure gases produced by combustion applies direct force to some component of the engine, such as pistons, turbine blades, or a nozzle. This force moves the component over a distance, generating useful mechanical energy.

The term *internal combustion* engine usually refers to an engine in which combustion is intermittent, such as the more familiar four-stroke and two-stroke piston engines, along with variants, such as the Wankel rotary engine. A second class of internal combustion engines use continuous combustion: gas turbines, jet engines and most rocket engines, each of which are internal combustion engines on the same principle as previously described.

A large number of different designs for ICEs have been developed and built, with a variety of different strengths and weaknesses. Powered by an energy-dense fuel (which is very frequently petrol, a liquid derived from fossil fuels), the ICE delivers an excellent power-to-weight ratio with few disadvantages. While there have been and still are many stationary applications, the real strength of internal combustion engines is in mobile applications and they dominate as a power supply for cars, aircraft, and boats, from the smallest to the largest.

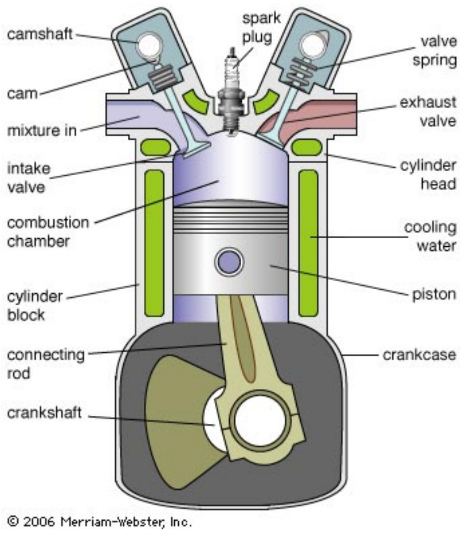


Figure 39: Elements of the Otto motor

The operation of the *four-stroke* ICE (or Otto motor) consists of four basic steps that repeat with every two revolutions of the engine:

1. **Intake** Combustible mixtures are emplaced in the combustion chamber

2. **Compression** The mixtures are placed under pressure

3. **Combustion (Power)** The mixture is burnt, almost invariably a deflagration, although a few systems involve detonation. The hot mixture is expanded, pressing on and moving parts of the engine and performing useful work.

4. **Exhaust** The combustion products are exhausted into the atmosphere.

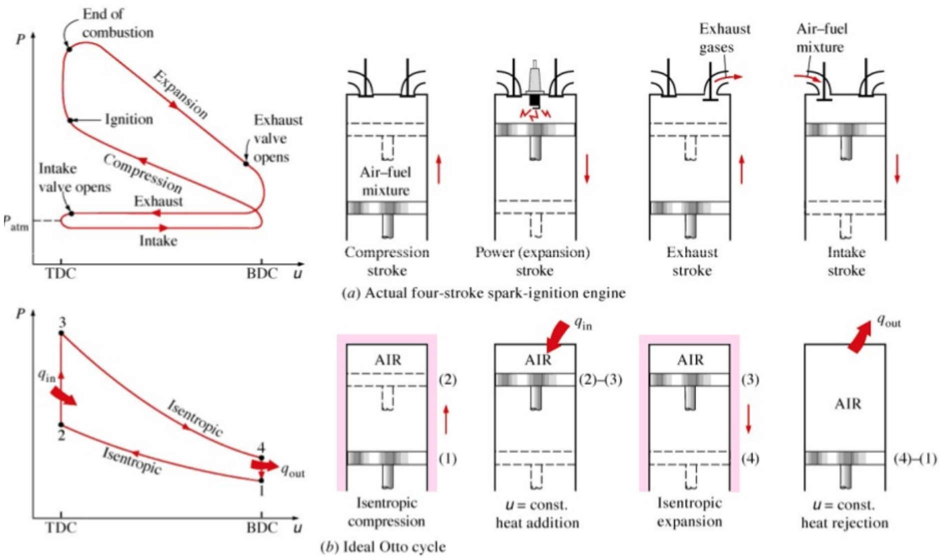


Figure 40: Indicator diagram of an ICE. Above is the real while below is the ideal indicator diagram.

5.3 Rankine cycle (steam engines)

A Rankine cycle describes a model of steam-operated heat engine most commonly found in power generation plants. Common heat sources for power plants using the Rankine cycle are the combustion of coal, natural gas and oil, also nuclear fission.

There are four processes in the Rankine cycle:

- Process 1-2: The working fluid is pumped from low to high pressure, as the fluid is a liquid at this stage the pump requires little input energy.
- Process 2-3: The high pressure liquid enters a boiler where it is heated at constant pressure by an external heat source to become a dry saturated vapour.
- Process 3-4: The dry saturated vapour expands through a turbine, generating power. This decreases the temperature and pressure of the vapour. It is very important that condensation shall not occur here, because even microscopic water droplets would damage the turbine blades which are rotating at very high speeds.
- Process 4-1: The wet vapour then enters a condenser where it is condensed at a constant pressure to become a saturated liquid.

In an ideal Rankine cycle the pump and turbine would be isentropic, i.e., the pump and turbine would generate no loss and hence maximize the net work output.

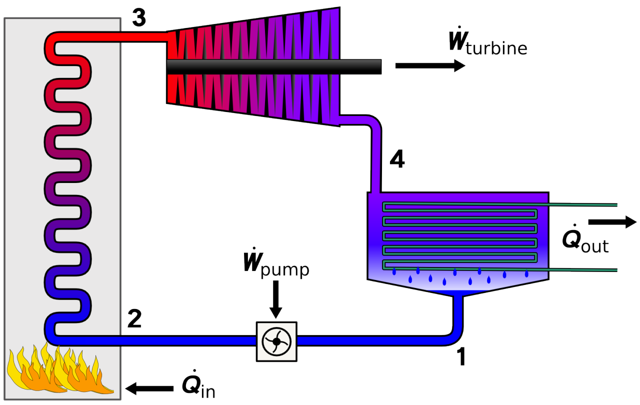


Figure 41: Rankine cycle scheme

The power balance of the cycle is as follows. Let h denote the *enthalpy* of the fluid

$$h = u + \frac{p}{\rho} + \cancel{\frac{v^2}{2}} + \cancel{gh}, \tag{89}$$

where the cancelled terms will be neglected as they are small compared to the the internal energy u and pressure work p/ρ. The internal energy u is

$$u = c_v T, \tag{90}$$

where $c_v[J/(kgK)]$ is the specific heat capacity of the fluid measured at constant volume (for water at ambient pressure at t=15^{o}C, $c_p = 4186 J/(kgK)$). The heat $\dot{Q}[J/kg]$ given to the fluid by the boiler changes the enthalpy of the fluid of mass flow rate $\dot{m}$. This equals to the heat obtained by burning $\dot{m}_{fuel}$ mass flow rate fuel with heat content $H[J/kg]$:

$$\dot{Q} = \dot{m}(h_3 - h_2) = \dot{m}_{fuel} H. \tag{91}$$

The turbine extracts the output energy by decreasing the enthalpy of the fluid:

$$\dot{W}_{turbine} = P_{turbine} = \dot{m}(h_3 - h_4), \tag{92}$$

and the condenser also extracts heat from the fluid (which is not necessary a power loss, can be used for secondary purposes e.g. for district heating networks):

$$\dot{Q}_{out} = \dot{m}(h_4 - h_1). \tag{93}$$

The pumping power $\dot{W}_{pump}$ is around the 1% of the turbine work output, thus it can be neglected.

5.4 Problems

Problem 5.1 During the measurement of a furnace the following data became available. The steam produced per hour is $\dot{m}_s = 6000$kg/h, while its enthalpy is $h_s = 3200$kJ/kg. The temperature of the feeding water is $t_w = 80$°C and the consumption of the coal ($H = 13.5$MJ/kg) is $\dot{m}_c = 1.6$tons/h. Calculate the efficiency of the furnace if the specific heat capacity of the water is $c_p = 4.2$kJ/(kgK) and the zero level of enthalpy is at 0°C! ($\eta = 79.6\%$)

Problem 5.2 In a steam boiler $\dot{m}_s = 2000$kg/h steam is produced with a pressure of $p_s = 10$bar and a temperature of $t_s = 250$°C from feeding water with a temperature of $t_w = 16$°C. The enthalpy of the steam is $h_s = 2940$kJ/kg, where the zero level of the enthalpy is given at 0°C and the specific heat capacity of the water is $c_p = 4.2$kJ/(kgK). The efficiency of the steam boiler is $\eta = 76\%$. Calculate the mass of coal ($H = 12$MJ/kg) needed to be burnt per hour! ($\dot{m}_c = 630$kg/h) During a reformation, the coal furnace is replaced by an oil furnace. With that change the efficiency of the steam boiler rises up to $\eta = 80\%$. Calculate the volume of the oil tank needed for a one-day-long operation if the heat content of the oil is $H = 40$MJ/kg and its density is $\rho = 950$kg/m^3! ($V = 4.536$m^3)

Worked problem 5.3 The sketch of a heat power plant can be seen in Fig. 42. The power of the electric generator is $P_{eg} = 50$MW, while its efficiency is $\eta_{eg} = 97\%$. The efficiency of the steam turbine that drives the generator is $\eta_{st} = 87\%$. The enthalpy of the freshly produced steam is $h_3 = 3370$kJ/kg, while its pressure and temperature are $p_3 = 110$bar and $t_3 = 500°$C respectively. The pressure in the condensator is $p_4 = 0.04$bar, while the enthalpy of the steam in it is $h_4 = 2180$kJ/kg.

- Calculate the mass flow rate of the steam needed to generate the given output power of the generator!
  $\dot{m}_s = \frac{P_{st,input}}{\Delta h} = \frac{P_{eg}}{\eta_{eg}\eta_{st}(h_3-h_4)} = 49.79\text{kg/s} = 179.2\text{tons/h}$

- Find the coal consumption of the steam boiler in tons/hour if its efficiency is $\eta_{sb} = 80\%$, the enthalpy of the feed water is $h_2 = 125$kJ/kg, and the heat content of the coal is $H = 20$MJ/kg!
  $\eta_{sb} = \frac{P_{sb,output}}{P_{sb,input}} = \frac{(h_3-h_2)\dot{m}_s}{\dot{m}_c H} \rightarrow \dot{m}_c = \frac{(h_3-h_2)\dot{m}_s}{\eta_{sb} H} = 36.344\text{tons/h}$

- Calculate the efficiency of the heat power plant!
  $\eta = \frac{P_{eg}}{P_{sb,input}} = \frac{P_{eg}}{\dot{m}_c H} = 24.8\%$

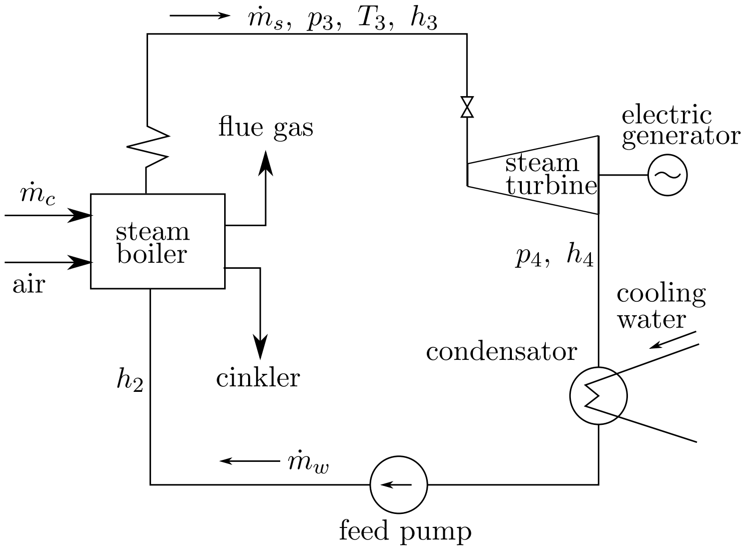
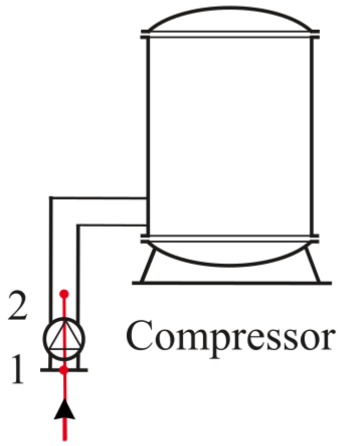


Figure 42: Sketch of the heat power plant in Problem 5.3

Problem 5.4

We convey air with a compressor. The mass flow rate of the air is $\dot{m} = 3.1\ \frac{\text{kg}}{\text{s}}$. The area of both the suction and pressure side is $A = 0.01227\ \text{m}^2$. The suction side is denoted with index (1), and the pressure side is denoted with (2). The pressures and temperatures at the pressure and suction side are the following: $t_1 = 30°C$, $t_2 = 221,85°C$, $p_1 = 1$ bar, $p_2 = 3$ bar. We can approximate the air as an ideal gas, therefore the ideal gas law holds: $\frac{p\ (\text{Pa})}{\rho\ \left(\frac{\text{kg}}{\text{m}^3}\right)} = R\left(\frac{\text{J}}{\text{kg}\cdot\text{K}}\right)\cdot T\ (\text{K})$. The specific gas constant is $R = 286\ \frac{\text{J}}{\text{kg}\cdot\text{K}}$, and the specific heat capacity at a constant volume is $c_v = 717\ \frac{\text{J}}{\text{kg}\cdot\text{K}}$. The suction and pressure side are above $h_1 = 30$ cm and $h_2 = 70$ cm the base level, respectively.



Calculate the numerical values in the table below! Analyse which terms in the enthalpy calculation are important, and which can be neglected!

No.	$\rho\ \left(\frac{\text{kg}}{\text{m}^3}\right)$	$q\ \left(\frac{\text{m}^3}{\text{s}}\right)$	$v\ \left(\frac{\text{m}}{\text{s}}\right)$	$v^2/2\ \left(\frac{\text{kJ}}{\text{kg}}\right)$	$gh\ \left(\frac{\text{kJ}}{\text{kg}}\right)$	$\frac{p}{\rho}\ \left(\frac{\text{kJ}}{\text{kg}}\right)$	$c_vT\ \left(\frac{\text{kJ}}{\text{kg}}\right)$	$h_t\ \left(\frac{\text{kJ}}{\text{kg}}\right)$
1	1.153	2.688	219.0	23.98	2.94e-03	86.7	217.4	328.0
2	2.119	1.463	119.2	7.11	6.87e-03	141.6	354.9	503.6
diff.	-	-	-	-16.88	3.92e-03	54.87	137.6	175.6
$\frac{\text{diff}}{\text{diff}(h_t)}$(%)	-	-	-	-9.61	2.24e-03	31.26	78.4	100.0

Problem 5.5 The volumetric flow rate of a pump, which conveys water is $q = 0.02305\ \frac{\text{m}^3}{\text{s}}$. The pressure at the suction side is 1.1 bar, and the pressure at the pressure side is $p_2 = 30$ bar. The temperature of the water at the suction side is $t = 17°C$, and the water is heated by due to the friction of the fluid in the pump by $\Delta T = 0.05°C$. The height of the suction side and the pressure side above the ground is $h_1 = 10$ cm and $h_1 = 50$ cm, respectively. The diameter of both the suction and pressure side pipes is $D = 50$ mm. The specific internal energy of the water at the pressure and suction side is $e_1 = 71.353$ kJ/kg and $e_2 = 71.249$ kJ/kg, respectively. The diameter of the pipe at the suction and pressure side is the same, so the kinetic energy change of the fluid can be neglected. The density of the water is $\rho = 1000\ \frac{\text{kg}}{\text{s}}$.

Calculate the numerical values in the table below! Analyse which terms in the enthalpy calculation are important, and which can be neglected. (The data highlighted with yellow is given.)

No.	v $\left(\frac{\text{m}}{\text{s}}\right)$	$v^2/2$ $\left(\frac{\text{kJ}}{\text{kg}}\right)$	gh $\left(\frac{\text{kJ}}{\text{kg}}\right)$	$\frac{p}{\rho}$ $\left(\frac{\text{kJ}}{\text{kg}}\right)$	e $\left(\frac{\text{kJ}}{\text{kg}}\right)$	h_t $\left(\frac{\text{kJ}}{\text{kg}}\right)$
1	11.74	6.891e-02	9.810e-04	1.100e-01	71.353	71.5
2	11.74	6.891e-02	4.905e-03	3.000e+00	71.416	74.5
diff.	-	0.00	3.924e-03	2.890	6.288e-02	2.957
$\frac{\text{diff}}{\text{diff}(h_t)}$(%)	-	0.00	0.133	97.74	2.127	100.0

Problem 5.6 An internal combustion engine consumes $m = 0.011$ kg during $t = 45$ s. The heating value of the fuel is $H = 43.5\ \frac{\text{MJ}}{\text{kg}}$, and the density of the fuel is $\rho = 737\ \frac{\text{kg}}{\text{m}^3}$. The useful mechanical power of the engine is $P_u = 2.1$ kW.

(a) Find the mass flow rate of the fuel. ($\dot{m} = 0.88\ \frac{\text{kg}}{\text{h}}$)

(b) Find the input power. ($P_{in} = 10.633$ kW)

(c) Find the specific fuel consumption. (SFC $= 0.1975\ \frac{\text{kg}}{\text{MJ}}$)

(d) Find the efficiency. ($\eta = 19.75\%$)

Problem 5.7 The configuration of a power plant is depicted in figure 41. The mass flow rate of the steam is $\dot{m}_s = 8.7\ \frac{\text{t}}{h}$. The pressure in the furnace and before the turbine is $p_2 = p_3 = 100$ bar. The heating value of the coal burned in the furnace is $H = 13.5\ \frac{\text{MJ}}{\text{kg}}$. The efficiency of the furnace is approximately $\eta_f = 100\%$. The density, temperature and specific enthalpy of the water before the furnace is $\rho_2 = 1002.689\ \frac{\text{kg}}{\text{m}^3}$, $t_2 = 20°C$ and $h_2 = 93.286\ \frac{\text{kJ}}{\text{kg}}$. The properties of the overheated steam after the furnace, and before the turbine are: $t_3 = 500°C$, $\rho_3 = 30.4758\ \frac{\text{kg}}{\text{m}^3}$, $h_3 = 3375.06\ \frac{\text{kJ}}{\text{kg}}$. The efficiency of the steam turbine is $\eta_t = 80\%$. The pressure and the temperature of the still overheated stream after the turbine $p_4 = 40$ mbar and $t_4 = 110°C$, and the density and the enthalpy are $\rho_4 = 0.022631\ \frac{\text{kg}}{\text{m}^3}$ and $h_4 = 2707.145\ \frac{\text{kJ}}{\text{kg}}$. In the heat exchanger the pressure is the same as after the turbine, and after the heat exchanger the temperature, density and the enthalpy of the water are $t_1 = 20°C$, $\rho_1 = 998.16\ \frac{\text{kg}}{\text{m}^3}$ and $h_1 = 83.921\ \frac{\text{kJ}}{\text{kg}}$.

(a) Find the specific enthalpy change in the furnace. ($\Delta h_f = 3281.8\ \frac{\text{kJ}}{\text{kg}}$)

(b) Find the power of the furnace. ($P_f = \dot{Q} = 7.931$ MW)

(c) Find the mass flow rate of the fuel. ($\dot{m}_f = 2.115 \ \frac{\text{t}}{h}$)

(d) Find the specific enthalpy change in the turbine. ($\Delta h_t = 667.91 \ \frac{\text{kJ}}{\text{kg}}$)

(e) Find the power of the turbine. ($P_t = 1.291$ MW)

(f) Find the specific enthalpy change in the condenser. ($\Delta h_c = 2623.2 \ \frac{\text{kJ}}{\text{kg}}$)

(g) Find the power of the condenser! ($P_c = 6.340$ MW)

(h) Find the efficiency of the electricity production. (enthalpy change end power of the pump are: $\Delta h_p = 9.365 \ \frac{\text{kJ}}{\text{kg}}$, $P_p = 0.0226$ MW; the efficiency is $\eta_e = 16.24\%$)

(i) Find the efficiency of the work cycle, of the heat in the condenser is used for heating up the water in a district heating system. ($\eta_c = 95.94\%$)

6 Unsteady operation of machines with constant acceleration

6.1 Introduction

Up to this point, we assumed that the forces (or torques) acting on a body are in equilibrium. Newton's second law states that

$$\sum \mathbf{F} = m\mathbf{a}, \tag{94}$$

that is, the acceleration of a body is parallel and directly proportional to the net force F and inversely proportional to the mass m. For rotational motion, we have

$$\sum T = \theta\varepsilon \tag{95}$$

In the engineering practice we often come across cases in which the net force $\sum F$ is constant. It follows than that the acceleration is also constant, hence the velocity and the displacement are

$$v(t) = \int a dt = at + v_0 \quad \rightarrow \quad s(t) = \int v(t)dt = \frac{a}{2}t^2 + v_0 t + s_0. \tag{96}$$

In a similar way, one obtains the following formulae for the angular velocity and rotation angle:

$$\omega(t) = \int \varepsilon dt = \varepsilon t + \omega_0 \quad \rightarrow \quad \phi(t) = \int \omega(t)dt = \frac{\varepsilon}{2}t^2 + \omega_0 t + \phi_0. \tag{97}$$

In the above formulae v_0, s_0, ω_0 and ϕ_0 are the initial (at $t = 0$) velocity, displacement, angular velocity and angle, respectively.

6.2 Examples of motion with constant acceleration

Example: Acceleration on a inclined plane (slope)

Let us reconsider the case of a body placed onto an inclined surface, as in Section 3.5, but in the absence of external force F. Equation 94 is written in vectorial form, now let us decompose it to a tangential and a normal component equation:

$$ma_t = mg\sin\alpha - \underbrace{\mu N}_{F_f} \neq 0 \quad \text{and} \tag{98}$$

$$ma_n = mg\cos\alpha + N = 0. \tag{99}$$

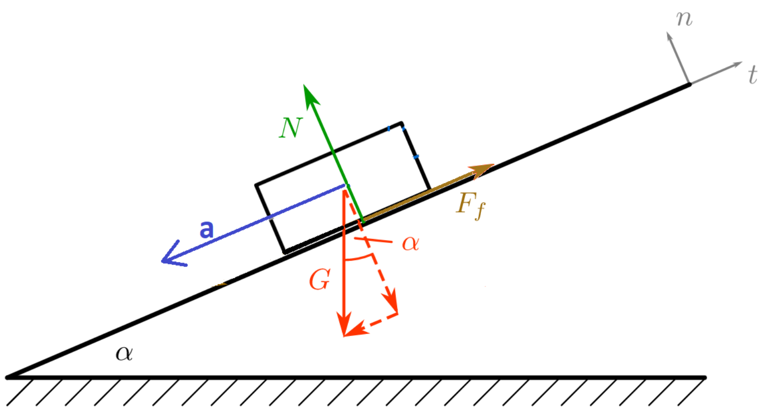


Figure 43: Accelerating object on an inclined plane

The normal acceleration is zero as the body moves parallel to the plate. Note that, on the other hand, as the friction force is unable to balance the tangential component of the gravitational force (remember that the value of μN is only the maximal possible value of the friction force!), the tangential acceleration is nonzero. The tangential acceleration is thus

$$a_t = g\left(\sin\alpha - \mu\cos\alpha\right). \tag{100}$$

Example: Braking of a rotating body

Let us consider a body with moment of inertia θ and initial revolution number $n_0 = \omega_0/(2\pi)$. We apply a constant braking force F_b on *two* brake drums, the friction coefficient is μ, the drums act on a radius R_b. We have

$$\theta\varepsilon = M_b = -2\mu F_b R_b = \text{const.} \neq 0 \tag{101}$$

The negative sign shows that we have deceleration. The angular velocity as a function of time is

$$\omega(t) = \omega_0 + \varepsilon t = \omega_0 - \frac{2\mu F_b R_b}{\theta}t, \tag{102}$$

thus the time needed for stopping the motion t_{stop} is when the angular velocity reaches zero:

$$\omega(t_{stop}) = 0 \rightarrow \tag{103}$$

$$t_{stop} = \frac{\omega_0\theta}{2\mu F_b R_b}. \tag{104}$$

The number of revolutions till the body stops is

$$N = \frac{\varphi(t_{stop})}{2\pi} = \frac{1}{2\pi}\left(\omega_0 t_{stop} - \frac{\varepsilon}{2}{t_{stop}}^2\right) \tag{105}$$

6.3 Problems

Problem 6.1 A train exhibits a traction force of 150 kN on a 600 m long path. Meanwhile, the velocity changes from 36 km/h to 54 km/h. The overall mass of the train is 1000 t. Find the change in the kinetic energy (62.5 MJ). Find the friction force (45.8 kN).

Problem 6.2 A car's velocity is 36 km/h, when it starts braking (all four wheels have brakes!). Assuming a friction coefficient of 0.6, find the diplacement needed for the stopping (8.5 m).

Problem 6.3 Due to the malfunction of the braking system, a train gets loose downwards a hill of 11 degree-slope at height 10 m. The rolling resistance is 0.08. The initial velocity was 10 m/s. Find the velocity at 0 m height (14.7 m/s).

Problem 6.4 The moment of inertia of the rotating part of a machine is 132 kgm^2. The initial revolution number is 750 rpm, the braking torque is 310 Nm. Find the time needed to stop the machine (33.4 s) and the number of revolutions (209 revs.).

Problem 6.5 The disc-like rotating part of a machine is of 0.5 m diameter and 250 kg. The initial revolution number is 1500 rpm, the friction coefficient in the bearings is 0.04, the radius of the bearing is 50mm (the radius on which the friction force acts). Find the time until the body stops if no braking force is applied (253 s). Find the braking torque needed if the element is to be stopped within 10 seconds (117.8 N).