Basic equations of flow systems, canonical form, characteristics, 1D unsteady subsonic flow, 2D supersonic steady flow

1. Basic flow equations of fluid flow

Notations:

w	velocity vector
p	pressure
ρ	density
T	absolute temperature
u	specific internal energy
e	specific total energy $e = u + \frac{w^2}{2}$
h	specific enthalpy
m	mass
I	momentum
F	force
E	energy
V	Volume
A	surface, area
n	normal vector of surface
P	power
$\dot{Q}$	heat power

Conservation of mass: $$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV + \int_A \rho \mathbf{w} \cdot \mathbf{n} dA = 0 .$$

Conservation of momentum: $$\frac{d\mathbf{I}}{dt} = \frac{d}{dt}\int_V \rho \mathbf{w} dV = \int_V \frac{\partial \rho \mathbf{w}}{\partial t} dV + \int_A \rho \mathbf{w}\mathbf{w} \cdot \mathbf{n} dA = \mathbf{F}_{body} + \mathbf{F}_{surface} .$$

Conservation of energy: $$\frac{dE}{dt} = \frac{d}{dt}\int_V \rho e dV + \int_A \rho e \mathbf{w} \cdot \mathbf{n} dA = P_{body} + P_{surface} + \dot{Q} .$$

By neglecting surface friction surface forces result from pressure distribution on channel walls.

$$\mathbf{F}_{pressure} = -\int_A p \cdot \mathbf{n} dA .$$

Its power is $$P_{pressure} = -\int_A p \mathbf{w} \cdot \mathbf{n} dA .$$

Let’s write these equations for a channel with cross section $A = A(x)$ varying along the x axis. The area A doesn’t change in time. Velocity **w** – average velocity– is parallel to the normal vector **n** of channel cross section A , thus $\mathbf{w} \cdot \mathbf{n} = w$.

Body forces will be neglected; surface forces will be identified with pressure force. The above equations are integrated on an elementary volume $dV = Adx$. We get three conservation equations in the above order.

Conservation of mass: $$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho A w)}{\partial x} = 0 ,$$

Conservation of momentum: $$\frac{\partial(\rho A w)}{\partial t} + \frac{\partial A\left(\rho w^2 + p\right)}{\partial x} = p\frac{dA}{dx} ,$$

Conservation of energy: $$\frac{\partial(\rho e A)}{\partial t} + \frac{\partial(\rho w e + p w)A}{\partial x} = \dot{Q} .$$

These equations can be written in the concise form $\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{\Phi}}{\partial x}=\mathbf{S}$, where

$$\mathbf{U}=\begin{pmatrix}\rho A\\ \rho wA\\ \rho eA\end{pmatrix},\qquad \mathbf{\Phi}=\begin{pmatrix}\rho wA\\ \left(\rho w^2+p\right)A\\ \left(\rho we+pw\right)A\end{pmatrix}\qquad \mathbf{S}=\begin{pmatrix}0\\ p\frac{dA}{dx}\\ \dot{Q}\end{pmatrix}.$$

Some numerical codes use this style as fluxes Φ are defined on the boundaries of elementary cells, e.g. hexahedra's while elements of U are defined in the centres of cells when integrating the conservation equations on a cell having a finite volume.
Through a lengthy computation one dimensional equations can be transformed into non-conservation form if e.g. conservation of mass is substituted into conservation of momentum after differentiating product-terms. We get finally

Conservation of mass: $$\frac{\partial \rho}{\partial t}+\rho\frac{\partial w}{\partial x}+w\frac{\partial \rho}{\partial x}=-\frac{\rho w^2}{A}\frac{dA}{dx}, \qquad (*)$$

Conservation of momentum: $$\frac{\partial w}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}+w\frac{\partial w}{\partial x}=0, \qquad (**)$$

Conservation of energy: $$\frac{\partial p}{\partial t}+w\frac{\partial p}{\partial x}+\rho c^2\frac{\partial w}{\partial x}=\dot{Q}-\frac{\rho wc^2}{A}\frac{dA}{dx}.$$

Speed of sound has been denoted by c in the last equation defined by $c^2=\frac{dp}{d\rho}=\kappa RT=\kappa\frac{p}{\rho}$.

We can summarize the three basic lows again this time in vector-matrix form:

$$\frac{\partial \mathbf{V}}{\partial t}+\mathbf{M}\frac{\partial \mathbf{V}}{\partial x}=\mathbf{S}.$$

Actually

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ w\\ p\end{pmatrix}+\begin{pmatrix}w & \rho & 0\\ 0 & w & \frac{1}{\rho}\\ 0 & \rho c^2 & w\end{pmatrix}\cdot\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ w\\ p\end{pmatrix}=\begin{pmatrix}-\frac{\rho w^2}{A}\frac{dA}{dx}\\ 0\\ \dot{Q}-\frac{\rho wc^2}{A}\frac{dA}{dx}\end{pmatrix}. \qquad (***)$$

The second term is the product of a matrix and a vector. Eigenvalues of matrix $\mathbf{M}=\begin{pmatrix}w & \rho & 0\\ 0 & w & \frac{1}{\rho}\\ 0 & \rho c^2 & w\end{pmatrix}$ play an important role with respect to existence of solutions of the equation system. The eigenvectors $\mathbf{e}_i$ are solutions of the homogenous algebraic equation $\left(\mathbf{M}-\lambda_i\mathbf{E}\right)\mathbf{e}_i=0$ where $\mathbf{E}$ is the unite matrix. In order that a solution exists $\det\left(\mathbf{M}-\lambda\mathbf{E}\right)=0$ must be valid. In details

$$\det\begin{vmatrix}w-\lambda & \rho & 0\\ 0 & w-\lambda & \frac{1}{\rho}\\ 0 & \rho c^2 & w-\lambda\end{vmatrix}=0,\text{ or } (w-\lambda)\left[(w-\lambda)^2-c^2\right]=0.$$

It can be seen that the three eigenvalues are

$$\lambda_1 = w,$$
$$\lambda_2 = w + c,$$
$$\lambda_3 = w - c.$$

The importance of these eigenvalues are obvious if we want to describe the unsteady isentropic flow of an ideal gas through a pipe of constant cross section. In this case equations (*) and (**) are

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial w}{\partial x} + w\frac{\partial \rho}{\partial x} = \frac{\partial p}{\partial t} + w\frac{\partial p}{\partial x} + \rho c^2\frac{\partial w}{\partial x} = 0 \tag{1}$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} + w\frac{\partial w}{\partial x} = 0 \tag{2}$$

we have introduced the square of sound velocity c in equation (*) giving the second form of (1).

Differentiating (1) with respect to t and (2) with respect to x then substituting $\frac{\partial^2 w}{\partial x^2}$ from the x-derivative of equation (1) we get a second order partial differential equation (PDE) for the unknown p. In this new equation only first order partial derivatives of the dependent variables stand beside the second partial derivatives of p:

$$\left(w^2 - c^2\right)\frac{\partial^2 p}{\partial x^2} + 2w\frac{\partial^2 p}{\partial x \partial t} + \frac{\partial^2 p}{\partial t^2} = F\left(w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial p}{\partial x}\right). \tag{3}$$

The above equation can be written in general form

$$a(x,t)\frac{\partial^2 p}{\partial x^2} + 2b(x,t)\frac{\partial^2 p}{\partial x \partial t} + c(x,t)\frac{\partial^2 p}{\partial t^2} = F(\text{lower order terms}). \tag{3*}$$

2. The type of the 2nd order PDE, canonical form

Equations (3*) is called a **second order PDE** for the unknown *p(x,t)* **quasilinear in its main part**. The terms containing second order derivatives are called the main part.
Coefficients a, b, c of the second order partial derivatives can be arranged into a symmetric matrix. This matrix has the form

in the general case (3*) $\mathbf{A} = \begin{pmatrix} a(x,t) & b(x,t) \\ b(x,t) & c(x,t) \end{pmatrix}$ and in the special case (3) $\mathbf{A} = \begin{pmatrix} w^2 - c^2 & w \\ w & 1 \end{pmatrix}$.

The type of the PDE is determined by the **sign** of the determinant $\det(\mathbf{A}) = a(x,t)c(x,t) - b^2(x,t)$ of matrix **A**.

- If det(**A**) > 0, ***definite***, the type of the PDE is **elliptic**,
- if det(**A**) = 0, ***semidefinite***, the type of the PDE is **parabolic**,
- if det(**A**) < 0, ***indefinite***, the type of the PDE is **hyperbolic.**

Let's transform the independent variables *x, t* through the functions $\xi = \xi(x, t)$, $\eta = \eta(x, t)$ leading to a simpler form of the equation. We suppose that ξ and η are continuously differentiable functions thus the Jacobian must be **nonzero**, $J = \frac{\partial(\xi,\eta)}{\partial(x,t)} \neq 0$. By applying the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\xi_x + \frac{\partial}{\partial \eta}\eta_x \quad ; \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\xi_t + \frac{\partial}{\partial \eta}\eta_t ,$$

further

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial^2}{\partial \xi^2}\xi_x + \frac{\partial^2}{\partial \xi \partial \eta}\eta_x\right)\xi_x + \frac{\partial}{\partial \xi}\xi_{xx} + \left(\frac{\partial^2}{\partial \eta \partial \xi}\xi_x + \frac{\partial^2}{\partial \eta^2}\eta_x\right)\eta_x + \frac{\partial}{\partial \eta}\eta_{xx} =$$

$$= \frac{\partial^2}{\partial \xi^2}\xi_x^{\,2} + 2\frac{\partial^2}{\partial \xi \partial \eta}\xi_x\eta_x + \frac{\partial^2}{\partial \eta^2}\eta_x^{\,2} + \xi_{xx}\frac{\partial}{\partial \xi} + \eta_{xx}\frac{\partial}{\partial \eta} .$$

Similarly

$$\frac{\partial^2}{\partial x \partial t} = \frac{\partial^2}{\partial \xi^2}\xi_x\xi_t + \frac{\partial^2}{\partial \xi \partial \eta}(\xi_x\eta_t + \eta_x\xi_t) + \frac{\partial^2}{\partial \eta^2}\eta_x\eta_t + \ldots ,$$

and

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \xi^2}\xi_t^{\,2} + 2\frac{\partial^2}{\partial \xi \partial \eta}\xi_t\eta_t + \frac{\partial^2}{\partial \eta^2}\eta_t^{\,2} + \ldots .$$

Putting these derivatives into the left hand side of Eq. (3*) the second order terms will be

$$\left(a(x,t)\xi_x^{\,2} + 2b(x,t)\xi_x\xi_t + c(x,t)\xi_t^{\,2}\right)\frac{\partial^2}{\partial \xi^2} +$$

$$+ 2\left(a(x,t)\xi_x\eta_x + b(x,t)(\xi_x\eta_t + \eta_x\xi_t) + c(x,t)\xi_t\eta_t\right)\frac{\partial^2}{\partial \xi \partial \eta} +$$

$$+ \left(a(x,t)\eta_x^{\,2} + 2b(x,t)\eta_x\eta_t + c(x,t)\eta_t^{\,2}\right)\frac{\partial^2}{\partial \eta^2} = \ldots . \qquad (4^*)$$

The coefficients of the first and third term have the same structure with the only difference that the coefficient in the first term contains derivatives of the new variable ξ while the third term derivatives of η. When equating the coefficients of the first and third term in Eq. (4) with zero only the middle term remains.

$$a(x,t)\xi_x^{\,2} + 2b(x,t)\xi_x\xi_t + c(x,t)\xi_t^{\,2} \overset{!}{=} 0 ,$$

$$a(x,t)\eta_x^{\,2} + 2b(x,t)\eta_x\eta_t + c(x,t)\eta_t^{\,2} \overset{!}{=} 0 \qquad (5^*)$$

From the first equation

$$\xi_t = \frac{-2b(x,t)\xi_x \pm \sqrt{4b^2(x,t)\xi_x^2 - 4c(x,t)a(x,t)\xi_x^2}}{2c(x,t)} = \frac{-b(x,t) \pm \sqrt{-\det(\mathbf{A})}}{c(x,t)}\xi_x .$$

The difference equation of a $\xi(x,t)$ = constant level as the change (total derivative) of ξ along such a line is zero:

$$d\xi = \xi_x dx + \xi_t dt = 0 , \quad \text{or} \quad \xi_t = -\frac{dx}{dt}\xi_x .$$

By comparing the last two equations one can see that the slope of a $\xi(x,t)$ = constant line is

$$\frac{dx}{dt}=\frac{b(x,t)\mp\sqrt{-\det(\mathbf{A})}}{c(x,t)}.$$

This equation has

- two real solutions if det(**A**) < 0 thus for hyperbolic equations,
- one real solution if det(**A**) = 0 thus for parabolic equations and
- no real solutions if det(**A**) > 0 thus for elliptic equations.

These characteristics had been found by transforming the 2nd order PDE into its "**canonical form**". The transformation was executed on the independent variables.

In the special case (3) the value of the determinant $\det(\mathbf{A})=\left(w^2-c^2\right)\cdot 1-w\cdot w=-c^2$ is negative our equation (3) is of hyperbolic type.
As we see; wave equation is typically hyperbolic.
The equation of heat conduction or of 2D diffusion ($\frac{\partial C}{\partial t}-\alpha\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)=-u\frac{\partial C}{\partial x}-v\frac{\partial C}{\partial y}$) is parabolic,

2D vortex free flow of an incompressible fluid ($\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0$) or an electromagnetic field is elliptic.

Equations (5^*) related to the special case are

$$(w^2-c^2)\xi_x{}^2+2w\xi_x\xi_t+\xi_t{}^2=0,$$
$$(w^2-c^2)\eta_x{}^2+2w\eta_x\eta_t+\eta_t{}^2=0 \qquad (5)$$

From the first equation

$$\frac{dx}{dt}=w\mp\sqrt{-\det(\mathbf{A})}=w\mp c.$$

The characteristic differential equation will disintegrate into a pair of two simple equations

$$\frac{dx}{dt}=w\pm c. \qquad (6)$$

The solutions of (6) are:

$$x=(w+c)t+\xi$$
$$x=(w-c)t+\eta$$

where rearranging for the integration constants ξ and η give

$$\xi=x-(w+c)t,$$
$$\eta=x-(w-c)t.$$

The slopes of the two series of characteristic lines: ξ = constant., and η = constant on the $t-x$ plane is identical with the eigenvalues of matrix **M**, but **NOT of A!!!**

After having found the transformation simplifying the main part of the 2nd order PDE we can calculate this transformed main part.

The coefficient of the derivative $\frac{\partial^2 p}{\partial\xi\partial\eta}$ in formula (4^*) will be $2\left(a(x,t)-\frac{b^2(x,t)}{c(x,t)}\right)\xi_x\eta_x$

and in the special case (4) this coefficient is $-4c^2$ thus the simple main part is

$$-4c^2\frac{\partial^2 p}{\partial\xi\,\partial\eta}=....,$$

this is the canonical form of the PDE, its type is ***hyperbolic***.

Through a second transformation $\psi = \xi + \eta, \varphi = \xi - \eta$ we get the other canonical form of hyperbolic 2nd order PDE's: $\frac{\partial^2 p}{\partial\xi\,\partial\eta} = \frac{\partial^2 p}{\partial\psi^2} - \frac{\partial^2 p}{\partial\varphi^2} = ...$

Method of characteristics (MOC); the form of ODE's to be solved along the characteristics

Let's add Eq.(1) to the ρc-times of Eq. (2) then push the term ρc into the derivatives of w! We receive $\frac{\partial p}{\partial t} + \rho wc\frac{\partial w}{\partial t} + c\frac{\partial p}{\partial x} + c\frac{\partial \rho cw}{\partial x} + w\frac{\partial p}{\partial x} + w\frac{\partial \rho cw}{\partial x} = 0$. This can be written in a denser form: $\frac{\partial}{\partial t}(p + \rho wc) + (w + c)\frac{\partial}{\partial x}(p + \rho cw) = 0$. Remembering that Eq. (6) with the upper sign was $w + c = \frac{dx}{dt}$, we see that $\frac{\partial}{\partial t}(p + \rho wc) + \frac{dx}{dt}\frac{\partial}{\partial x}(p + \rho cw) = \frac{d}{dt}(p + \rho cw) = 0$ along the ξ=constant characteristics. Similarly, along the η = constant characteristics $\frac{d}{dt}(p - \rho cw) = 0$ is true.

Thus we have received two ODE's which can be integrated very easily and the unknowns p and w can be computed in small time steps.

Initial and boundary conditions

The importance of the type of 2nd order PDE-s lies in the fact that it determines the type of **initial and boundary conditions** assuring unique solutions.

If; for example; we prescribe for the starting time $t = 0$ along a section of the x-axis the initial values $p(x, t=0)$ then the function $p(x,t)$ can be computed for the triangular region between the two characteristics ξ=constant starting from the left border and η = constant starting from the right border of the x-interval.

Thus if we need to get the solution over the whole pipe length $0 \le x \le L$ then not only the initial $p(x, t=0)$ distribution but also the boundary values at $x = 0$ and at $x = L$ must be known.

These are the boundary values

$$p(x = 0, t) \quad \text{and} \quad p(x = L, t).$$

3. Steady flow problem leading to a hyperbolic equation

The 2D steady isentropic flow of a compressible gas is described by the following equations (u, v denote the components of velocity vector **w**):

Continuity:
$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0. \tag{7}$$

Eulerian equations
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x}, \tag{8}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial y}. \tag{9}$$

Adding the *u*-times of Eq. (8) to the *v*-times of Eq. (9) and substituting Eq. (7) into the RHS of this equation one receives

$$\left(u^2-c^2\right)\frac{\partial u}{\partial x}+\left(v^2-c^2\right)\frac{\partial v}{\partial y}+uv\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=0\,. \qquad (10)$$

Isentropic flow is also a vortex free flow this is a corollary of the *Crocco equation* **(*)** .

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0\,. \qquad (11)$$

(*) For the reversible, adiabatic case the first law of thermodynamics says $dq=Tds=dh-\frac{dp}{\rho}=0$. Thus $\mathrm{grad}\,h=\frac{1}{\rho}\mathrm{grad}\,p$. According to this and to Euler's equation $\mathrm{grad}\left(\frac{\mathbf{w}^2}{2}\right)-\mathbf{w}\times\mathrm{rot}\,\mathbf{w}=-\frac{1}{\rho}\mathrm{grad}\,p=-\mathrm{grad}\,h$. This gives $\mathbf{w}\times\mathrm{rot}\,\mathbf{w}=\mathrm{grad}\left(\frac{\mathbf{w}^2}{2}+h\right)=\mathrm{grad}\,h_{ö}$. However this is zero, thus either $\mathrm{rot}\,\mathbf{w}=0$ or $\mathrm{rot}\,\mathbf{w}$ is parallel with the velocity vector **w** (Beltrami flow).

Now we shall define the coordinate *x*, the main flow direction as "time-like". Then Eqs. (10) and (11) can be rewritten in a matrix vector formulation. Partial derivatives will be written in the short "subscript-form".

$$\begin{pmatrix}u^2-c^2 & uv\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}u_x\\ v_x\end{pmatrix}+\begin{pmatrix}uv & v^2-c^2\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}u_y\\ v_y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

With concise notation: $\mathbf{A}_1\mathbf{w}_x+\mathbf{A}_2\mathbf{w}_y=\mathbf{0}$.

We need the earlier (***) form thus we look for $\mathbf{w}_x+\mathbf{A}_1^{-1}\cdot\mathbf{A}_2\mathbf{w}_y=\mathbf{0}$. We must find the eigenvalues of the matrix $\mathbf{M}=\mathbf{A}_1^{-1}\cdot\mathbf{A}_2$. Those eigenvalues will determine the slopes of characteristics running through the *x-y* plane. The inverse of matrix $\mathbf{A}_1$ denoted by $\mathbf{A}_1^{-1}$ is the transposed matrix constructed from the under determinants belonging the elements of $\mathbf{A}_1$ divided by the determinant of $\mathbf{A}_1$. For a 2x2 matrix it is very easy to find. $\det\begin{pmatrix}u^2-c^2 & uv\\ 0 & -1\end{pmatrix}=-\left(u^2-c^2\right)$,

thus $\mathbf{A}_1^{-1}=\frac{-1}{\left(u^2-c^2\right)}\cdot\begin{pmatrix}-1 & uv\\ 0 & u^2-c^2\end{pmatrix}=\begin{pmatrix}\frac{1}{u^2-c^2} & \frac{uv}{u^2-c^2}\\ 0 & -1\end{pmatrix}$. From this

$$\mathbf{M}=\mathbf{A}_1^{-1}\cdot\mathbf{A}_2=\begin{pmatrix}\frac{1}{u^2-c^2} & \frac{uv}{u^2-c^2}\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}uv & v^2-c^2\\ 1 & 0\end{pmatrix}=\begin{pmatrix}\frac{2uv}{u^2-c^2} & \frac{v^2-c^2}{u^2-c^2}\\ -1 & 0\end{pmatrix}. \qquad (12)$$

The eigenvalues of matrix **M** are solutions of $\det(\mathbf{M}-\lambda\mathbf{E})=0$. One can control that these are $\lambda_1=\frac{uv+\alpha c^2}{u^2-v^2}$ and $\lambda_2=\frac{uv-\alpha c^2}{u^2-v^2}$, resp. Here $\alpha\overset{denotes}{=}\sqrt{\frac{u^2+v^2-c^2}{c^2}}\sqrt{\frac{|\mathbf{w}|^2-c^2}{c^2}}=\sqrt{M^2-1}$, where *M* is the local Mach-number.

After a lengthy calculation we get an alternate form of the eigenvalues:

$$\lambda_1 = \frac{u+\alpha v}{\alpha u - v} \qquad \text{and} \qquad \lambda_1 = \frac{\alpha v - u}{\alpha u + v}. \tag{13}$$

With further notations $\frac{v}{u} = \tan\vartheta$ and $\frac{1}{\alpha} = \tan\beta$ (14)

$$\lambda_1 = \frac{\frac{1}{\alpha} + \frac{v}{u}}{1 - \frac{v}{u}\cdot\frac{1}{\alpha}} = \frac{\tan\beta + \tan\vartheta}{1 - \tan\vartheta\cdot\tan\beta} = \tan(\vartheta+\beta).$$

Thus $\lambda_1 = \tan(\vartheta+\beta)$ and similarly $\lambda_2 = \tan(\vartheta-\beta)$. The slopes of characteristics is equal to the eigenvalues (see the earlier marked sentence). Now one can draw the velocity vector **w** and the two characteristics ξ = constant and η = constant in the *x*-*y* plane.

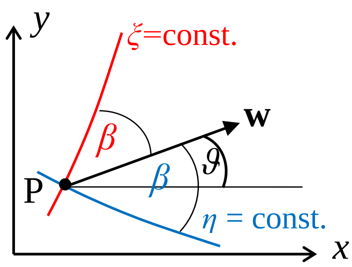

Fig. 1 Characteristics ξ, η, velocity vector **w**, angles β and ϑ

The information con of point P is bordered by the characteristic line ξ = constant and η = constant starting from point P. Happenings only in this region are influenced by point P. These characteristic lines are also called as Mach-lines.

Equations to be solved along the characteristics

From $\frac{v}{u} = \tan\vartheta$ follows $u = w\cos\vartheta$; $v = w\sin\vartheta$. Naturally, the partial derivatives can be expressed with w and ϑ, e.g. $\frac{\partial u}{\partial x} = \frac{\partial w}{\partial x}\cdot\cos\vartheta - w\cdot\sin\vartheta\frac{\partial \vartheta}{\partial x}$. Other first order derivatives have similar forms. After substituting these into Eq. (10) and rearranging we have

$$c^2 w\left(\frac{\partial\vartheta}{\partial x}\sin\vartheta - \frac{\partial\vartheta}{\partial y}\cos\vartheta\right) + \left(w^2 - c^2\right)\left(\frac{\partial w}{\partial x}\cos\vartheta + \frac{\partial w}{\partial y}\sin\vartheta\right) = 0. \tag{10a}$$

From the definition of α, β and from the LHS formula (14) $\sin\beta = \frac{c}{w}$, $\cos\beta = \frac{\sqrt{w^2-c^2}}{w}$, thus $\cot\beta = \frac{\sqrt{w^2-c^2}}{c} = \sqrt{M^2-1}$. With all these and after dividing (10a) with cw^2 we get:

$$\sin\beta\left(\frac{\partial\vartheta}{\partial x}\sin\vartheta - \frac{\partial\vartheta}{\partial y}\cos\vartheta\right) + \frac{\cot\beta}{w}\cos\beta\left(\frac{\partial w}{\partial x}\cos\vartheta + \frac{\partial w}{\partial y}\sin\vartheta\right) = 0.$$

After a similar process the cosβ-times of Eq. (11) is:

$$\cos\beta\left(\frac{\partial\vartheta}{\partial x}\cos\vartheta + \frac{\partial\vartheta}{\partial y}\sin\vartheta\right) + \frac{1}{w}\cos\beta\left(\frac{\partial w}{\partial x}\sin\vartheta - \frac{\partial w}{\partial y}\cos\vartheta\right) = 0$$

Finally adding these equations (and subtracting in a second step)

$$\cos(\vartheta-\beta)\frac{\partial\vartheta}{\partial x}+\sin(\vartheta-\beta)\frac{\partial\vartheta}{\partial y}+\frac{\cot\beta}{w}\left(\cos(\vartheta-\beta)\frac{\partial w}{\partial x}+\sin(\vartheta-\beta)\frac{\partial w}{\partial y}\right)=0\text{, or rearranged}$$

$$\underline{\underline{\cos(\vartheta-\beta)}}\left(\underline{\underline{\frac{\partial\vartheta}{\partial x}}}+\frac{\cot\beta}{w}\frac{\partial w}{\partial x}\right)+\underline{\underline{\sin(\vartheta-\beta)}}\left(\underline{\underline{\frac{\partial\vartheta}{\partial y}}}+\frac{\cot\beta}{w}\frac{\partial w}{\partial y}\right)=0. \tag{15}$$

Along the characteristics η = constant $\lambda_2=\tan(\vartheta-\beta)=\dfrac{\sin(\vartheta-\beta)}{\cos(\vartheta-\beta)}=\left.\dfrac{dy}{dx}\right|_{\eta=\text{áll.}}$, so

$$\underline{\underline{\frac{\partial}{\partial\xi}}}=\frac{\partial}{\partial x}\cdot\left.\frac{\partial x}{\partial\xi}\right|_{\eta=\text{áll.}}+\frac{\partial}{\partial y}\cdot\left.\frac{\partial y}{\partial\xi}\right|_{\eta=\text{áll.}}=\underline{\underline{\cos(\vartheta-\beta)\cdot\frac{\partial}{\partial x}+\sin(\vartheta-\beta)\cdot\frac{\partial}{\partial y}}}.$$

If we substitute equality $\cot\beta=\sqrt{M^2-1}$ into (15) we receive

$$\underline{\underline{\frac{\partial\vartheta}{\partial\xi}}}+\frac{\sqrt{M^2-1}}{w}\frac{\partial w}{\partial\xi}=0 \qquad\text{and}\qquad \frac{\partial\vartheta}{\partial\eta}-\frac{\sqrt{M^2-1}}{w}\frac{\partial w}{\partial\eta}=0\text{, resp.} \tag{16}$$

By the help of Fig. 2 we show how to compute intersection P of characteristics starting from points L and R:

$$\frac{y_P-y_R}{x_P-x_R}=\tan(\vartheta_R+\beta_R)\text{ and }\frac{y_P-y_L}{x_P-x_L}=\tan(\vartheta_L-\beta_L)\text{, resp.;}$$

from these

$x_P=\dfrac{x_R\cdot\tan(\vartheta_R+\beta_R)-x_L\cdot\tan(\vartheta_L-\beta_L)-y_R+y_L}{\tan(\vartheta_R+\beta_R)-\tan(\vartheta_L-\beta_L)}$, followed by y_P expressed from one of the above equations.

The system of equations to be solved numerically is:

$\underline{\underline{\vartheta_p-\vartheta_R}}+\sqrt{M_R^2-1}\dfrac{w_P-w_R}{w_R}=0$ on ξ=const. and $\underline{\underline{\vartheta_p-\vartheta_L}}+\sqrt{M_L^2-1}\dfrac{w_P-w_L}{w_L}=0$ on η=const. lines.

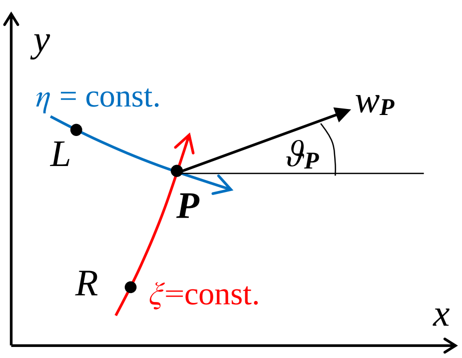

Fig. 2 Flow computation in point ***P*** from known values in points *L* and *R*