Open channel flow – weir rupture

A long rectangular channel of constant cross section is divided into two parts by a thin weir. The longitudinal coordinate is denoted by x. The weir is located at x = 0. In the left (L) and right (R) channel parts the water depths are $y(x<0, t<0) = y_L$ and $y(x>0, t<0) = y_R$ resp. The water velocity is zero $v(x<0, t<0) = 0 = v_L$, $v(x>0, t<0) = v_R = 0$ in both channel parts for $t < 0$. The width of the channel may be set to b = 1m.
At t = 0 the weir is suddenly bursting. A flood wave will travel to right and the water depth will fall in the left channel part. Water will stream from left to right, thus the stream velocity will rise in the middle region but it remains zero in the undisturbed parts. The flood wave is an abrupt change in water depths; the falling wave has a continuous shape.

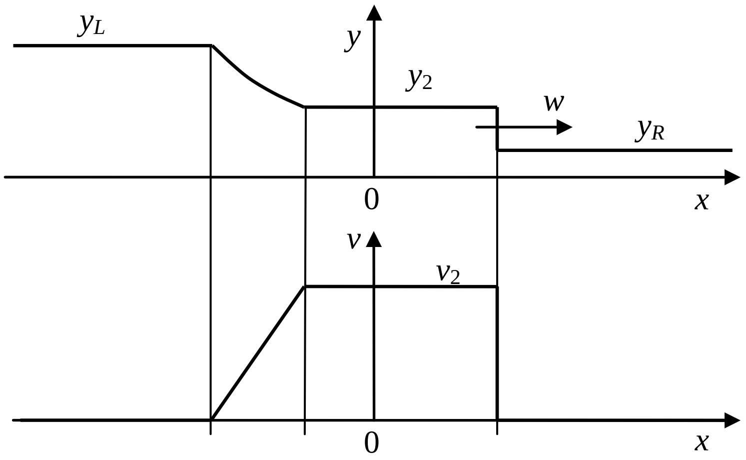

Fig.1 Water depth and velocity after weir rupture

1. Flood wave **Az itt következők magyarul olvashatók álló vízugrásra a "Vízugrás-lökéshullám" előadásvázlatban**

The flood wave can be approximated by a water jump of finite height difference but zero extension. This flood wave travels with velocity w to the right. On the right hand side of the flood wave the water depth remains y_R, left of the wave the water depth is y_2 (see the figure above). By considering a moving control volume around the water jump the flow will be stationary. We can write the conservation of volume and momentum for the control volume.

Continuity: $$-1\cdot y_R w = 1\cdot y_2 (v_2 - w). \quad (1)$$

Conservation of momentum: $$-\rho\cdot 1\cdot y_2 (v_2 - w)^2 - \rho\cdot 1\cdot y_R (-w)^2 + \rho g\left(\frac{y_2}{2}1\cdot y_2 - \frac{y_R}{2}1\cdot y_R\right) = 0. \quad (2)$$

Introducing the abbreviation $Y = \frac{y_2}{y_R}$ for the depth-ratio across the water jump Eq. (1) gives

$$v_2 = w\frac{Y-1}{Y}. \quad (3)$$

Similarly from Eq. (2) we get $$v_2 = \frac{g y_R}{2w}\left(Y^2 - 1\right). \quad (4)$$

Equating these and dropping (Y-1) $\frac{g y_R}{2w}(Y+1) = \frac{w}{Y}$. Here $g y_R = a_R^2$, a_R being the wave celerity in the right channel part. Solving for Y the well known formula for a stationary water jump results:

$$Y=\frac{-1+\sqrt{1+8Fr_w^2}}{2} \quad \text{with} \quad Fr_w^2=\frac{w^2}{a_R^2}.$$

2. Region 2 between the fan and the flood wave

The water fall in the wave moving to the left occurs in the form of a fan (see Fig. 2). As the wave celerity in region L is higher than in region 2 the top of this wave travels faster than its bottom, the wave gets more and more flat during its movement. The fan is bordered by two C$^-$ characteristics, their tangents in the x-t plane are $C_L^-: \left.\frac{dx}{dt}\right|_L^- = v_L - a_L$ and $C_L^-: \left.\frac{dx}{dt}\right|_2^- = v_2 - a_2$. Here the wave speeds $a_R=\sqrt{gy_R}$ and $a_2=\sqrt{gy_2}$ are used.

C_1^+ characteristics run in the region 1 of the fan between the two borders. From the St-Venant form of the MOC v_1+2a_1 is constant along the C^+ characteristics. Thus:

$v_L+2a_L=v_2+2a_2=v_2+2\sqrt{gy_2}=v_2+2\sqrt{gy_R\frac{y_2}{y_R}}=v_2+2a_R\sqrt{Y}$. However in our case $v_L=0$ thus

$\frac{a_L}{a_R}=\frac{\sqrt{gy_L}}{\sqrt{gy_R}}=\frac{v_2}{2a_R}+\sqrt{Y}=\frac{v_2}{2\sqrt{gy_R}}+\sqrt{Y}$. From this v_2 can be expressed:

$$v_2=\left(\sqrt{\frac{y_L}{y_R}}-\sqrt{Y}\right)2\sqrt{gy_R}. \tag{5}$$

Eqs. (3)-(5) are three equations for the three unknowns v_2, w and Y containing the depth ratio parameter $\frac{y_L}{y_R}$. A convergent iteration procedure in form (6) may be found. As a starting value Y=1 can be set.

$$Y=\frac{\left(\frac{y_L}{y_R}+Y-\frac{(Y-1)^2(Y+1)}{8Y}\right)^2}{4\frac{y_L}{y_R}} \tag{6}$$

A good power function fit is

$$Y\cong 0{,}9482\cdot\left(\frac{y_L}{y_R}\right)^{0{,}6164} \tag{6/a}$$

in the range $1{,}2\le\frac{y_L}{y_R}\le 10$ with R^2 = 0,9994. For given y_R values v_2 and w can be approximated too using Eq. (5) then Eq. (3).

3. Fan of falling water

Along a C_L^+ characteristics entering from region L into the fan

$$v_L+2a_L=v_1+2a_1=2a_L \tag{7}$$

as v_L = 0. Thus

$$v_1+2a_1=\text{constant}. \tag{8}$$

Along a C_1^- characteristics defined by $\left.\frac{dx}{dt}\right|_1^- = v_1 - a_1$ starting from the origin and passing through the fan

$$v_1 - 2a_1 = \text{constant}. \tag{9}$$

Adding and subtracting Eqs. (8) and (9) we find that $v_1 = \text{constant}$ and $a_1 = \text{constant}$ separately. This means that $\left.\frac{dx}{dt}\right|_1^- = v_1 - a_1 = \text{constant}$ or

$$v_1 - a_1 = \frac{x}{t}, \tag{10}$$

the C_1^- characteristics are straight lines. Substituting a_1 from Eq.(7) $\frac{x}{t} = v_1 - a_1 = v_1 + \frac{v_1}{2} - a_L = \frac{3}{2}v_1 - a_L$. This is the solution for the water velocity in the fan:

$$v_1 = \frac{2}{3}\left(a_L + \frac{x}{t}\right). \tag{11}$$

From (10)

$$a_1 = \frac{3}{2}a_L - \frac{1}{3}\frac{x}{t}. \tag{12}$$

Naturally the water depth is given by

$$y_1 = \frac{a_1^2}{g}. \tag{13}$$

In equations (11) and (12) the interval of validity for $\frac{x}{t}$ is given by $-a_L \le \frac{x}{t} \le \frac{3}{2}v_2 - a_L$. This completes the solution for the fan. Finally the Froude number distribution can also be computed.

4. Critical water depth ratio

Programming above equations and running the program with different $\frac{y_L}{y_R}$ ratios one can observe that the inclination of the right border C_2^- of the fan can be negative or positive. It is interesting to find the critical $\frac{y_L}{y_R}$ ratio for which the right border is identical with the *t*-axis (*x*=0 for all right border points of the fan).

From $\left.\frac{dx}{dt}\right|_2^- = v_2 - a_2 \overset{!}{=} 0$ we see that $v_2 = a_2 = \sqrt{gy_2} = \sqrt{g \cdot Yy_R} = \sqrt{Y} \cdot \sqrt{gy_R}$. Equating this with (5) gives

$$\sqrt{Y} \cdot \sqrt{gy_R} = \left(\sqrt{\frac{y_L}{y_R}} - \sqrt{Y}\right) 2\sqrt{gy_R} \quad \text{or} \quad \sqrt{\frac{y_L}{y_R}} = \frac{3}{2}\sqrt{Y}. \tag{14}$$

Putting this into (6) gives after some calculations a qubic equation for the critical value of Y: $Y^3 - 3Y^2 - Y + 1 = 0$. The physically relevant solution is Y_{cr} = 3,21432. Finally from (14)

$$\left.\frac{y_L}{y_R}\right|_{cr} = \frac{9}{4}Y_{cr} = 7{,}2322 \tag{15}$$

5 Net of characteristic lines

For a subcritical water depth ratio the C^+ and C^- characteristic lines, the fan and the water jump location are shown below.

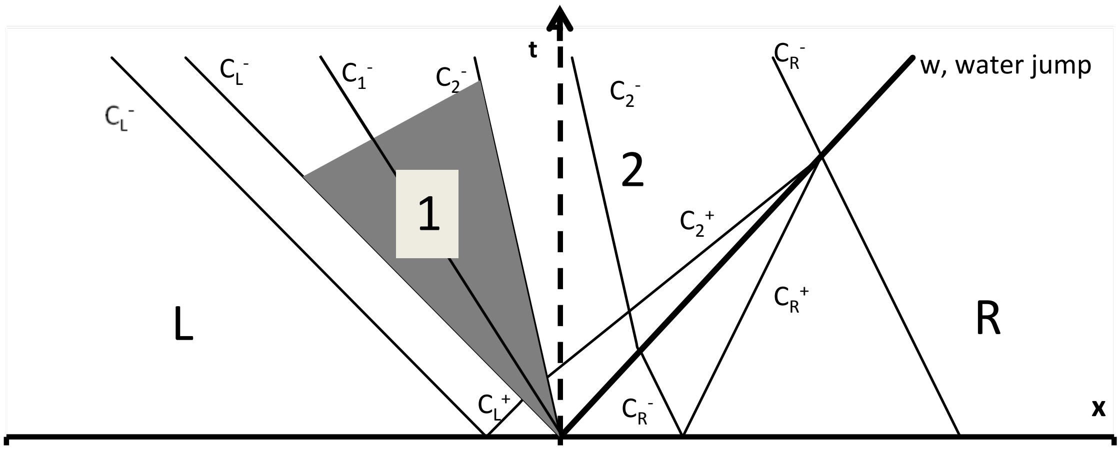

Fig. 2 Characteristic lines for sudden weir rupture