Unsteady Flow in Pipe Networks

lecure notes

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1 A few numerical techniques in a nutshell

1.1 Solving systems of algebraic equtions

Consider the problem of finding the solution of

$$\underline{f}(\underline{x}) = \underline{0}, \tag{1}$$

where f is some (complicated) nonlinear function.

In the 1D case, Newton's technique (or sometime called Netwon-Raphson technique) improves the previous solution x_n by finding the intersection of the x axes and the tangent line of the function evaluated at x_n:

$$f'(x_n) \approx \frac{\Delta y}{\Delta x} = \frac{f(x_n)}{x_n - x_{n+1}} \quad \rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

In the multidimensional case, we have

$$\underline{x}_{n+1} = \underline{x}_n - \left(\underline{f}'(\underline{x}_n)\right)^{-1} \underline{f}(\underline{x}_n). \tag{3}$$

Newton's method is an extremely powerful technique, in general the convergence is quadratic: as the method converges on the root, the difference between the root and the approximation is squared (the number of accurate digits roughly doubles) at each step. However, there are some difficulties with the method.

Difficulty in calculating derivative of a function Newton's method requires that the derivative is calculated directly. An analytical expression for the derivative may not be easily obtainable and could be expensive to evaluate. In these situations, it may be appropriate to approximate the derivative by using the slope of a line through two nearby points on the function, e.g.

$$f'(x_n) \approx \frac{f(x_n + \Delta x) - f(x_n)}{\Delta x}, \quad \text{with e.g.} \quad \Delta x_n = 0.01 x_n. \tag{4}$$

Failure of the method to converge to the root Sometimes the technique runs into an infinte loop. In such cases relaxation may help, i.e. we require the method to improve the solution only partially:

$$x_{n+1} = x_n - \omega \frac{f(x_n)}{f'(x_n)}, \quad \text{with} \quad 0 \leq \omega \leq 1. \tag{5}$$

Overshoot If the first derivative is not well behaved in the neighborhood of a particular root, the method may overshoot, and diverge from that root. Furthermore, if a stationary point of the function is encountered, the derivative is zero and the method will terminate due to division by zero. Use relaxation in such cases.

Poor initial estimate A large error in the initial estimate can contribute to non-convergence of the algorithm.

Mitigation of non-convergence In a robust implementation of Newton's method, it is common to place limits on the number of iterations, bound the solution to an interval known to contain the root, and combine the method with a more robust root finding method.

Slow convergence for roots of multiplicity > 1 If the root being sought has multiplicity greater than one, the convergence rate is merely linear (errors reduced by a constant factor at each step) unless special steps are taken. When there are two or more roots that are close together then it may take many iterations before the iterates get close enough to one of them for the quadratic convergence to be apparent. However, if the multiplicity m of the root is known, one can use the following modified algorithm that preserves the quadratic convergence rate:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{6}$$

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Note that Newton's technique is directly applicable to complex-valued functions. An interesting and beautiful experiment is to color the complex plane based on the basin of attraction of the roots of a complex polinomial function, say, $x^5 - 1 = 0$, giving rise to fractals.

The code below gives an example of Newton's method in 1D. Do some experiments with the relaxation factor `relax`.

Newton's method in 1D

```
function newtonsmethod

x=1; max_err=1e-4; max_iter=100; relax=0.5; iter=0;

f=funtosolve(x);
while abs(f)>max_err && iter<max_iter
    fprintf('\n iter=%2d, x=%7.5f, |f|=%5.3e',iter,x,abs(f));
    x=x-relax*f/derivative(x);
    f=funtosolve(x);
    iter=iter+1;
end
fprintf('\n iter=%2d, x=%7.5f, |f|=%5.3e',iter,x,abs(f));
end

function y=funtosolve(x)
y=cos(x)-x^3;
end

function dydx=derivative(x)
```

```
dydx=-sin(x)-3*x^2;
end
```

This is another example of Newton's method, now in 2D. Note that now the initial guess `x0` is a *column* vector and instead of `abs(f)`, we are using `norm(f)`.

Newton's method in 2D

```
function newtonsmethod2d

x=[1 0]'; max_err=1e-4; max_iter=100; relax=1;

f=funtosolve(x);
iter=0;
while norm(f)>max_err && iter<max_iter
    fprintf('\n iter=%2d, x=[+%7.5f +%7.5f], |f|=%5.3e',iter,x,norm(f));
    x=x-relax*inv(jac(x))*f;
    f=funtosolve(x);
    iter=iter+1;
end
fprintf('\n iter=%2d, x=[+%7.5f +%7.5f], |f|=%5.3e',iter,x,norm(f));
end

function y=funtosolve(x)
y=[cos(x(1))-x(2)^3
    x(1)*sin(x(2))];
end

function dydx=jac(x)
dydx=[-sin(x(1)) -3*x(2)^2
      sin(x(2)) x(1)*cos(x(2))];
end
```

1.2 Estimating derivatives with finite differences

1.2.1 First derivatives

Consider the problem of computing the derivative of function $f(x)$ at some point. We fix a stepsize h, than we have several possibilities: Forward difference:

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} \tag{7}$$

Backward difference:

$$f'(x) \approx \frac{f(x)-f(x-h)}{h} \tag{8}$$

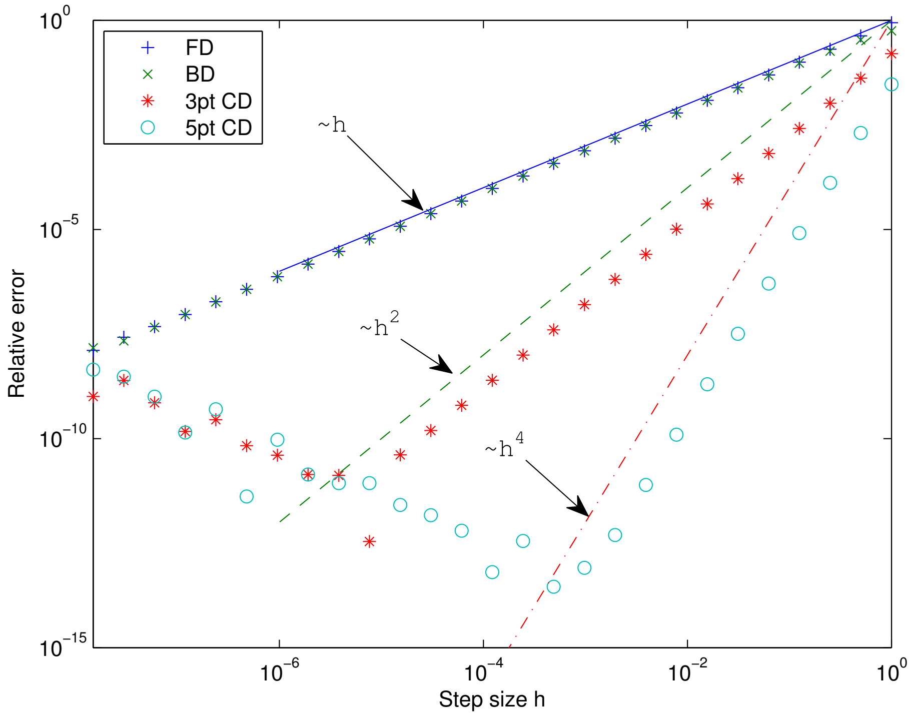


Figure 1: Accuracy of several finite difference schemes.

3-point central difference:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \tag{9}$$

5-point central difference:

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} \tag{10}$$

An important feature of a differentiation scheme is the 'speed' with which the approximation error e decreases as the step size h is decresed. An 'order m' scheme ensures that $e \propto h^m$, which is demonstrated in Figure 1. Note that one should not be greedy in terms of h as below some stepsize the limited accuracy of number representation (roundoff errors) start to spoil the accuracy of the scheme.

Numerical differentiation

```
function numdiff
figure(1), clf
h=1; x=1; dydx(1)=funder(x); e(1)=0;
for j=1:27
    dydx(2)=(f(x+h)-f(x))/h;
    dydx(3)=(f(x)-f(x-h))/h;
    dydx(4)=(f(x+h)-f(x-h))/2/h;
    dydx(5)=(-f(x+2*h)+8*f(x+h)-8*f(x-h)+f(x-2*h))/12/h;
```

```
    for i=2:length(dydx)
        e(i)=abs((dydx(1)-dydx(i))/dydx(1));
    end
    loglog(h,e(2),'+',h,e(3),'x',h,e(4),'*',h,e(5),'o'), hold on
h=h/2;
end
hh=[1 1e-6];
loglog(hh,hh,'-',hh,hh.^2,'--',hh,hh.^4,'-.'), hold off
xlabel('Step size h'), ylabel('Relative error')
legend('FD','BD','3pt CD','5pt CD',2), axis([0 1 1e-15 1])
end

function y=f(x)
y=sin(x);
end

function dydx=funder(x)
dydx=cos(x);
end
```

1.2.2 Second derivatives

The simplest possibility is to use the central difference scheme at the half grid points (second-order accuracy):

$$f''(x) \approx \frac{f'\left(x+\frac{h}{2}\right) - f'\left(x-\frac{h}{2}\right)}{\Delta h} = \frac{1}{h}\left(\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}\right)$$
$$= \frac{f(x+h)-2f(x)+f(x-h)}{h^2} \tag{11}$$

Another scheme using five points providing fourth-order accuracy is

$$f''(x) \approx \frac{-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)}{12h^2}. \tag{12}$$

1.3 Solving ordinary differential equations

In general a system of ordinary differential equations (ODEs) have the form of

$$\underline{x}' = \underline{f}\left(\underline{x}, t\right), \tag{13}$$

where the function $\underline{x}(t)$ is to be determined. In the case when the function $\underline{f}$ is only dependent upon t implicitly, that is $\underline{f} = \underline{f}(\underline{x})$, the ODE is called *autonomous*, otherwise it is called *non-autonomous*. There are two main groups of problems that can be described by ODEs:

- Boundary value problems (BVPs), where the function values at the boundaries ($\underline{x}(a)$ and $\underline{x}(b)$) are prescribed. These are usually describing a problem where the independent variables are space coordinates.
- Initial value problems (IVPs), where the function values are given at an initial time ($\underline{x}(t_0)$). These are usually describing problems where the independent variable is time. (In this subject we mainly deal with IVPs.)

In most of the cases there is no known analytical solution for an IVP. In these cases one tries to find a numerical approximation for the solution function ($\underline{x}(t)$). All these numerical methods are common in the sense that they give an approximate formula for the time derivative ($\dot{\underline{x}}$) and calculate the approximate solution at distinct time steps ($\underline{x}_n(t_n)$). Here three basic methods will be presented briefly.

1.3.1 Explicit Euler method

The basic equation reads:

$$\dot{\underline{x}} \approx \frac{\underline{x}(t+\Delta t)-\underline{x}(t)}{\Delta t} = \frac{\underline{x}_{n+1}-\underline{x}_n}{\Delta t} = \underline{f}(\underline{x}_n, t_n), \tag{14}$$

thus in every new time step the new function value can be computed by the formula:

$$\underline{x}_{n+1} = \underline{x}_n + \Delta t \underline{f}(\underline{x}_n, t_n). \tag{15}$$

This method is explicit since the new function value ($\underline{x}_{n+1}$) can be explicitly expressed with the help of the previous values ($\underline{x}_n$), hence it is fast. On the other hand it is very unstable and inaccurate (first order method), so in general it is not used. One way to improve the method is to use implicit Euler method.

1.3.2 Implicit or backward Euler method

The basic equation reads:

$$\dot{\underline{x}} \approx \frac{\underline{x}_{n+1}-\underline{x}_n}{\Delta t} = \underline{f}(\underline{x}_{n+1}, t_{n+1}). \tag{16}$$

This method is implicit, since for every new time step one has to solve a system of algebraic equations to get $\underline{x}_{n+1}$. This can be performed for example by the previously presented Newton's method. One advantage of the method is its stability (see Section 1.3.4), however its accuracy is not improved (first order method). One way to develop the accuracy is to use a higher order scheme such as Runge-Kutta 4.

1.3.3 Runge-Kutta 4 method

The basic equation reads:

$$\underline{x}_{n+1} = \underline{x}_n + \Delta t \left[\frac{\underline{k}_1}{6} + \frac{\underline{k}_2}{3} + \frac{\underline{k}_3}{3} + \frac{\underline{k}_4}{6} \right], \tag{17}$$

with

$$\begin{aligned}
\underline{k}_1 &= \underline{f}\left(\underline{x}_n, t_n\right), \\
\underline{k}_2 &= \underline{f}\left(\underline{x}_n + \frac{\Delta t}{2}\underline{k}_1, t_n + \frac{\Delta t}{2}\right), \\
\underline{k}_3 &= \underline{f}\left(\underline{x}_n + \frac{\Delta t}{2}\underline{k}_2, t_n + \frac{\Delta t}{2}\right), \\
\underline{k}_4 &= \underline{f}\left(\underline{x}_n + \Delta t \underline{k}_3, t_n + \Delta t\right).
\end{aligned}$$

This method is fourth order as it is denoted in its name. This means a great improvement in its accuracy over the Euler method. Since it is an explicit scheme it is fast as well.

1.3.4 Stability

Stability is a very important property of the different kinds of ODE solvers. The common way of investigating a method's stability is to consider the following scalar ODE:

$$\dot{x} = \lambda x, \quad \text{where} \quad \lambda \in \mathbb{C} \quad \text{and} \quad \Re(\lambda) < 0. \tag{18}$$

The condition $\Re(\lambda) < 0$ is required since we want to deal with a 'normal physical phenomenon' that is stable and only the numerical method can introduce instability in the solution. If we apply the different schemes to (18) we get the following geometric series:

- Explicit Euler method: $x_{n+1} = (1 + \Delta t\lambda)x_n$
- Implicit Euler method: $x_{n+1} = \frac{1}{1-\Delta t\lambda}x_n$
- Runge-Kutta 4 method: $x_{n+1} = \left(\Delta t\lambda + \frac{(\Delta t\lambda)^2}{2!} + \frac{(\Delta t\lambda)^3}{3!} + \frac{(\Delta t\lambda)^4}{4!}\right)x_n$

Since a geometric series is stable if $|q| < 1$, where q is the quotient, we have to investigate the coefficient of x_n. In Figure 2 the stability regions of the different ODE solvers are plotted. Since $\Re(\lambda) < 0$ and $\Delta t \in \mathbb{R}^+$, $\Re(\Delta t\lambda) < 0$, that is we are only interested in the left side of the complex plane. Δt should be chosen so that $\Delta t\lambda$ is in the stability regions. As it can be seen the implicit Euler method is unconditionally stable, while the stability region of the Runga-Kutta 4 method is wider than the explicit Euler's.

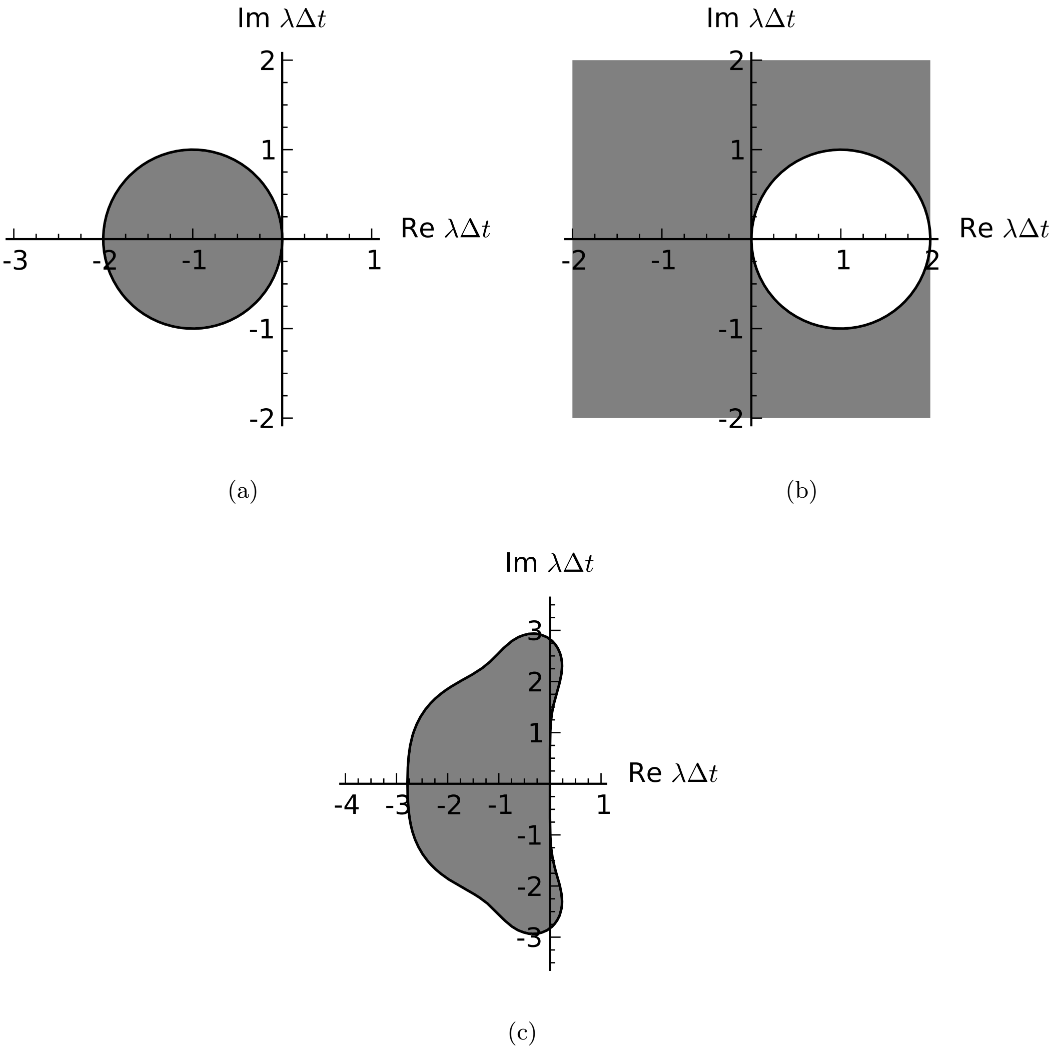


Figure 2: Stability charts of the explicit Euler (a), the implicit Euler (b) and the Runge-Kutta 4 (c) methods (stable regions are *gray*).

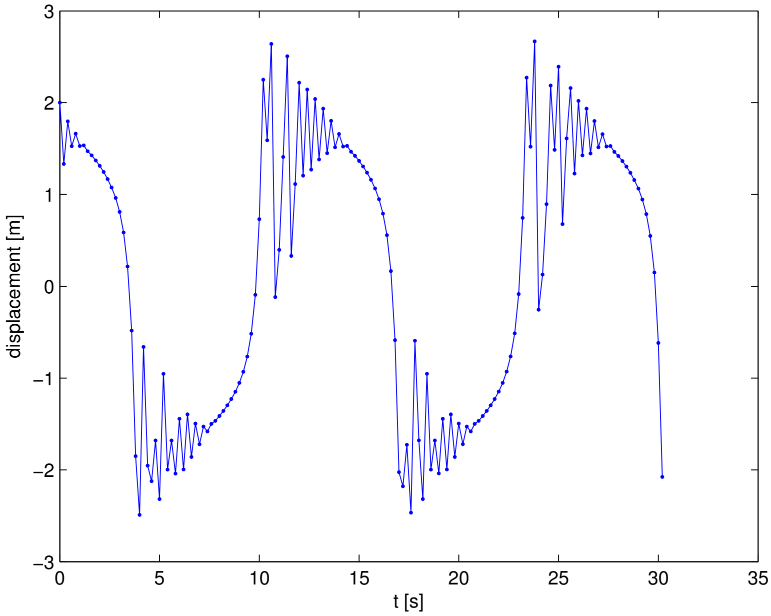
1.3.5 Accuracy

Another very important aspect of ODE solvers is accuracy. Basically there are two ways to improve accuracy: using a higher order method (such as Runge-Kutta 4) or using a so-called adaptive method (such as Runge-Kutta-Fehlberg 45). The basis of an adaptive method is to somehow estimate the error of each new step and adopt the time-step so that a prescribed error tolerance is kept. A very simple way to achieve that (although computationally very expensive) is to determine the new value by two different ways: perform one step with the time-step from the previous step (Δt) and stepping two steps with $(1/2)\Delta t$, then comparing the two values. If the difference of the two values are greater than a prescribed error, than the time-step is decreased and the method is repeated, otherwise the new value is accepted and we continue on. Such a method is implemented below for the explicit Euler scheme on the Van der Pol equation. The Van der Pol equation is often used to test adaptive solvers, since it is stiff, that is the solution function has very rapid variations. The output of the scheme is shown in Figure 3. (It is worth noting that an adaptive solver poses a solution to instability issues as well.)

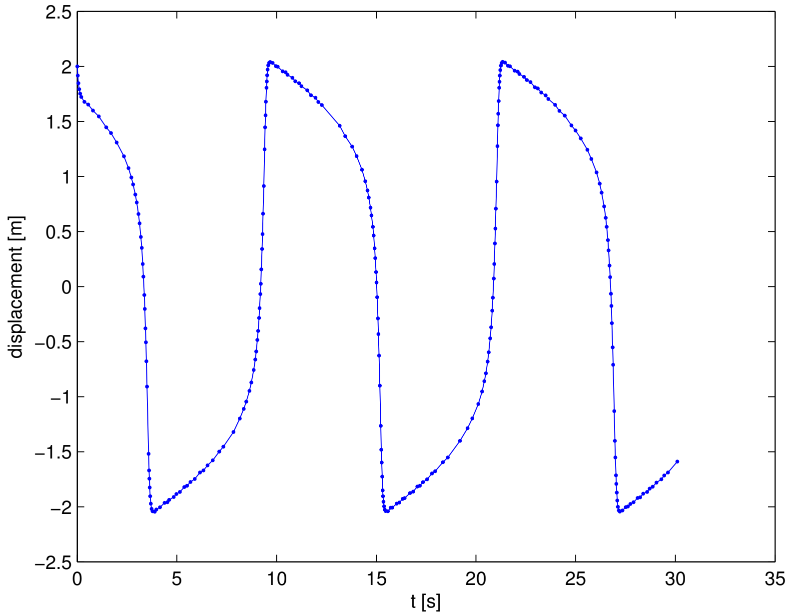
Adaptive explicit Euler method for the Van der Pol equation

```
function van_der_pol

max_err=1e-2;        % maximum allowed error
t0=0;                % initial time
dt=0.2;              % initial time step
tEnd=30;             % final time
x0=[2,0];            % initial condition
adap=1;              % adaptivity switcher
inc_fac=5;           % increment factor for dt

t=t0; x=x0; tVec=t0; xVec=x0;
if (adap==1)    % adaptivity on
    while (t < tEnd)
        xNew1=x+dt*ode_fun(x);      % one step with dt
        xNew2_1=x+dt/2*ode_fun(x);  % two steps with dt/2
        xNew2=xNew2_1+dt/2*ode_fun(xNew2_1);
        if (norm(xNew1-xNew2) < max_err)
            tVec=[tVec, t+dt]; xVec=[xVec; xNew1];
        else
            % halving dt till required error is reached
            while (norm(xNew1-xNew2) > max_err)
                dt=dt/2;
                xNew1=x+dt*ode_fun(x);
                xNew2_1=x+dt/2*ode_fun(x);
                xNew2=xNew2_1+dt/2*ode_fun(xNew2_1);
            end
            tVec=[tVec, t+dt]; xVec=[xVec; xNew1];
```

```
        end
        t=t+dt; x=xNew1; dt=dt*inc_fac;
    end
elseif (adap==0) % adaptivity off
    while (t < tEnd)
        xNew=x+dt*ode_fun(x);
        tVec=[tVec, t+dt]; xVec=[xVec; xNew];
        t=t+dt; x=xNew;
    end
end

plot(tVec,xVec(:,1),'.-'); xlabel('t [s]'); ylabel('displacement [m]')

function dxdt=ode_fun(x) % Van der Pol equation with mu=5
dxdt(1)=5*(x(1)-1/3*x(1)^3-x(2));
dxdt(2)=1/5*x(1);
```



(a)



(b)

Figure 3: Computed displacement of the Van der Pol equation with regular (a) and with adaptive (b) solvers

2 Calculation of the wave speed under different circumstances

The wave speed in the simplest case, *in circular thin walled pipes* is

$$a = \sqrt{\frac{E_{reduced}}{\rho}}, \quad \frac{1}{E_{reduced}} = \frac{1}{E_{fluid}} + \frac{D}{\delta E_{pipe}} \tag{19}$$

if the fluid streaming in the pipe is slightly compressible. Here ρ is the fluid density, D and δ are the inner pipe diameter and pipe wall thickness, E_{fluid} is the bulk modulus of the fluid, E_{pipe} is the elasticity modulus of the pipe wall material.

Compared to this basic situation the different *longitudinal supports or fixing* of the pipe will cause changes. The Poisson-number μ is the ratio of the longitudinal and circumferential stresses, its value for thin pipe wall is 0.5. The influence of the support can be considered by a factor n.

$$a = \sqrt{\frac{E_{reduced}}{\rho}}, \quad \frac{1}{E_{reduced}} = \frac{1}{E_{fluid}} + n\frac{D}{\delta E_{pipe}}. \tag{20}$$

For thin walled pipes $n = n_{thin}$ where

- axial support of the pipe at both ends: $n_{thin} = 1 - \mu/2$,
- longitudinal support throughout the pipe length: $n_{thin} = 1 - \mu^2$,
- without support, naturally: $n_{thin} = 1$.

If compared to the diameter the wall is thick the value of n will be calculated from the above value of n_{thin}:

$$n = \frac{2\delta}{D}(1+\mu) + \frac{D}{D+\delta} n_{thin}. \tag{21}$$

For gas flow

$$a = \sqrt{\frac{dp}{d\rho}}, \tag{22}$$

where assuming ideal gas $dp/d\rho = \kappa RT$, thus $a = \sqrt{\kappa RT} = \sqrt{\kappa p/\rho}$, which is the well-known isothermal speed of sound. As $E/\rho = a^2 = dp/d\rho = \kappa p/\rho$, obviously for gases

$$E_{gas} = \kappa p. \tag{23}$$

For open channel flow instead of $dp/d\rho$, the actual coefficient in the continuity equation is $dy/dA = 1/B$. Multiplying the equation

$$B\left(\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x}\right) + A\frac{\partial v}{\partial x} = 0 \tag{24}$$

with ρ/A and introducing the gravitational acceleration g:

$$\frac{B}{Ag}\left(\frac{\partial \rho g y}{\partial t}+v\frac{\partial \rho g y}{\partial x}\right)+\rho\frac{\partial v}{\partial x}=0. \tag{25}$$

This equation has the same dimension as the earlier continuity equations from which $B/(Ag)=1/a^2$, thus the free surface wave speed is

$$a=\sqrt{\frac{Ag}{B}}. \tag{26}$$

In air conditioning the ventilation channels often have rectangular cross section bended from thin metal plates. These, 1-2 m long channel segments are fixed to each other. Experiments with standing waves have proved that in this case the wave speed a strongly depends on the angular frequency ω of the pressure wave. This leads to its dispersion. The $a(\omega)$ function has the form:

$$a(\omega)=\frac{c}{\sqrt{1+\rho c^2 f(\omega)}}, \tag{27}$$

where c is the isentropic sound velocity in free air, the function $f(\omega)$ depends on the geometry and material properties of the channel. L is the dimension of the channel side, δ is the thickness of channel wall:

$$f(\omega)=\frac{2L^3}{E_{wall}\delta^3\Omega^5}\left(\frac{2}{\cot\Omega+\coth\Omega}-\Omega\right),\quad \text{and}\quad \Omega=\sqrt[4]{\frac{3\rho L^4\omega^2}{4E_{wall}\delta^2}}. \tag{28}$$

The wave speed in liquids strongly depends on *free gas content*. The wave speed in gaseous fluids can drop up to a few $10m/s$-s although the wave speed in pure normal air is $340m/s$. The mixture characterized by void fraction $\alpha=V_g/V$ is composed of free gas of density ρ_g and fluid of density ρ_f (naturally $1-\alpha=V_f/V$). The mass of the mixture is $\rho_g V_g+\rho_f V_f=\rho V$. Dividing this by the total volume V of the mixture and introducing the above abbreviation for the void fraction α the mean density is: $\rho=\alpha\rho_g+(1-\alpha)\rho_f$. Now the wave speed must be calculated from the mean density and reduced elasticity modulus. The elasticity will also depend on the compressibility of the gas. By the definition of the elasticity modulus (Hooks law) $dV_g=-V_g/E_g dp$, and $dV_f=-V_f/E_f dp$. The total change of volume V is

$$dV=dV_g+dV_f=-\left(\frac{\alpha V}{E_g}+\frac{(1-\alpha)V}{E_f}\right)dp=-\left(\frac{\alpha}{E_g}+\frac{(1-\alpha)}{E_f}\right)V dp \tag{29}$$

and thus the reduced elasticity modulus E_e is

$$E_e=-V\frac{dp}{dV}=\frac{1}{\left(\frac{\alpha}{E_g}+\frac{(1-\alpha)}{E_f}\right)}. \tag{30}$$

Finally the square of the wave speed (considering that the extension of the pipe wall compared to the compressibility of gas content is negligible):

$$a^2=\frac{E_e}{\rho}=\frac{1}{\left(\frac{\alpha}{E_g}+\frac{(1-\alpha)}{E_f}\right)(\alpha\rho_g+(1-\alpha)\rho_f)}=\frac{1}{\left(\frac{\alpha}{\kappa p}+\frac{(1-\alpha)}{E_f}\right)(\alpha\rho_g+(1-\alpha)\rho_f)}, \tag{31}$$

as $E_g = \kappa RT\rho = \kappa p$ (see above). This rather complicated formula can be simplified realizing that in the denominator in the first bracket the first, in the second bracket the second term is dominant

$$a = \sqrt{\frac{E_e}{\rho}} \approx \sqrt{\frac{\kappa p}{\alpha(1-\alpha)\rho_f}}. \tag{32}$$

Differentiating Eq. (32) with respect to α gives the minimum of the wave speed at $\alpha = 0.5$.

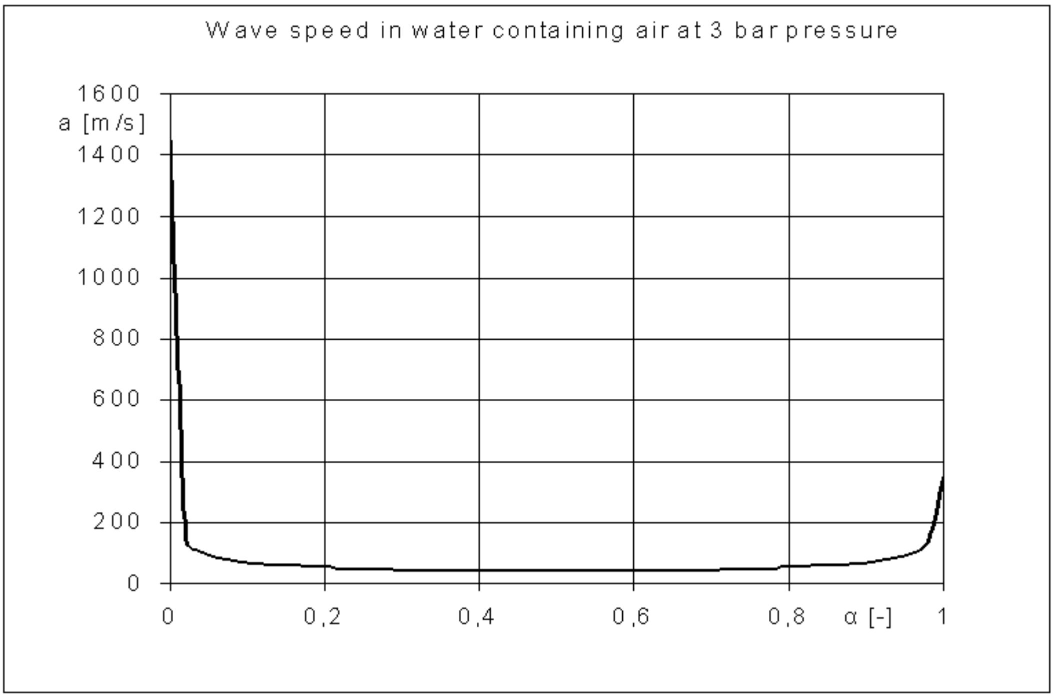


Figure 4: Wave speed in water containing air at 3 bar pressure

3 Model of a ventilation channel with rectangular cross-section

Euler-Bernoulli equation of the deflection of a beam (width: Δz, thickness: h) under uniform load in steady state:

$$EI_z \frac{d^4 w}{dx^4} = q \tag{33}$$

- E: elasticity modulus, $E = 2 \cdot 10^{11} Pa$ for steel
- I_z: area moment of inertia, $I_z = \frac{\Delta h^3}{12}$
- w: deflection of the longer channel side (u: deflection of the shorter channel side)
- x: longitudinal coordinate of the longer channel side (y: coordinate of the shorter channel side)

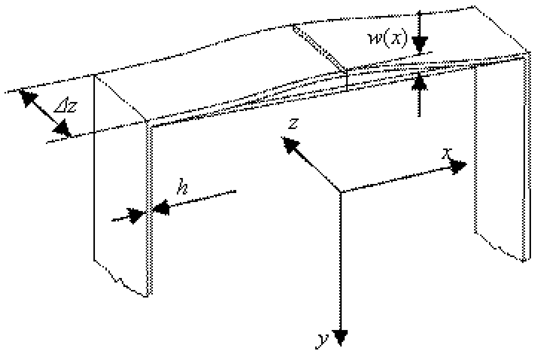


Figure 5: Wave speed in water containing air at 3 bar pressure

The above differential equation for *constant load* q (force per unit length, $q = dp\Delta z$ where dp is the change of overpressure in the channel) is:

$$EI_z \frac{d^4w}{dx^4} = dp\Delta z \quad \text{or} \quad \frac{d^4w}{dx^4} = \frac{dp\Delta z}{EI_z} \overset{\text{denoted}}{=} P. \tag{34}$$

The *general solution* of this differential equation is

$$w(x) = P\frac{x^4}{24} + C_3\frac{x^3}{6} + C_2\frac{x^2}{2} + C_1x + C_0 \tag{35}$$

As we suppose *symmetry with respect to the centreline* of the beam (at $x = 0$) the terms of uneven order will drop:

$$w(x) = P\frac{x^4}{24} + C_2\frac{x^2}{2} + C_0 \tag{36}$$

and similarly

$$u(x) = P\frac{y^4}{24} + D_2\frac{y^2}{2} + D_0 \tag{37}$$

for the shorter side of the channel.

3.1 Boundary conditions

The corners are fixed: $w(L/2) = 0$ and $u(-B/2) = 0$. The tangents of the channel sides at the corner are perpendicular as the angle of the corner keeps its value 90°: $w''(L/2) = u''(-B/2)$. Denoting the *side ratio* of the rectangular cross section by $s = B/L$ the solutions satisfying the boundary conditions are

$$w(x) = P\frac{x^4}{24} + \frac{PL^2}{24}\left(2s^2 - 2s - 1\right)\frac{x^2}{2} + \frac{PL^4}{8\cdot 24}\left(\frac{1}{2} + 2s - 2s^2\right), \tag{38}$$

$$u(y) = P\frac{y^4}{24} + \frac{PL^2}{24}\left(-s^2 - 2s + 2\right)\frac{y^2}{2} + \frac{PL^4}{8\cdot 24}s^2\left(-2 + 2s + \frac{s^2}{2}\right). \tag{39}$$

The area change dA of the cross section $A = BH$ is the sum of the integrals of the deflections of the for side walls:

$$dA = 2\int_{x=-\frac{L}{2}}^{x=\frac{L}{2}} w(x)dx + 2\int_{y=-\frac{yL}{2}}^{y=\frac{sL}{2}} w(y)dy = \frac{PL^5}{24}\frac{s^5 + 5\left(s^4 - s^3 - s^2 + s\right) + 1}{15}. \tag{40}$$

If both the gas is compressible and the channel area is changing under the pressure change the equation of continuity is

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho A v)}{\partial x} = \frac{\partial(\rho A)}{\partial t} + v\frac{\partial(\rho A)}{\partial x} + \rho A\frac{\partial v}{\partial x} = \frac{d(\rho A)}{dp}\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \rho A\frac{\partial v}{\partial x} = 0. \tag{41}$$

After some calculations and introducing the isentropic wave velocity c in the gas we have

$$\begin{aligned}\frac{1}{\rho A}\left(A\frac{d\rho}{dp} + \rho\frac{dA}{dp}\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial v}{\partial x} &= \left(\frac{1}{\rho c^2} + \frac{1}{A}\frac{dA}{dp}\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial v}{\partial x} \\ &\overset{\text{denoted}}{=} \left(\frac{1}{\rho c^2} + \Phi\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial v}{\partial x} = \left(\frac{1}{\rho a^2}\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial v}{\partial x} = 0. \end{aligned} \tag{42}$$

The wave velocity in the channel denoted above by a is

$$a = \frac{c}{\sqrt{1 + \rho c^2 \Phi}} \quad \text{with} \quad \Phi = \frac{1}{A}\frac{dA}{dp} = \frac{L^3}{15Eh^3}\frac{s^5 + 5\left(s^4 - s^3 - s^2 + s\right) + 1}{2s}, \tag{43}$$

where ρ is the density of air at the given air temperature. Here $a = 173,4m/s$ if the air density is $\rho = 1,2kg/m^3$ and $c = 340m/s$ with other parameters given at the end of this section.

Experiments have not proved these formulae. Why? Because the channel side walls have a mass and one has to consider the inertia of this mass. The equation of *dynamic bending* of a slender isotropic homogeneous beam of constant cross section under a constant transverse uniform load is

$$EI_z\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = q(t) \overset{!}{=} \Delta z \cdot \hat{p} \cdot e^{\imath\omega t}. \tag{44}$$

Here m is the mass per unit length, $m = \rho_c h\Delta z$, ρ_c is the density of the channel wall and $d\hat{p} \cdot e^{\imath\omega t}$ is a harmonic excitation in the form of a complex function. We look for the general solution of this partial differential equation, PDE – fourth order in space, second order in time – as the product of a function depending only on space and another function depending only on time (this method is called Fourier decomposition):

$$w = \hat{w}(x)e^{\imath\omega t}. \tag{45}$$

After differentiation with respect to x and t, respectively and substituting into the PDE

$$E\frac{\Delta z h^3}{12}\frac{d^4\hat{w}}{dx^4}e^{\imath\omega t} + \rho_c h\Delta z(\imath\omega)^2 e^{\imath\omega t} = \Delta z d\hat{p}e^{\imath\omega t} \tag{46}$$

by dropping the exponential function and Δz both occurring in all terms and noticing that the square of the imaginary unit $\imath$ is $\imath^2 = -1$ and finally multiplying by $12/(Eh^3)$ one gets:

$$\frac{d^4\hat{w}}{dx^4} - \frac{12\rho_c}{Eh^2}\hat{w}\omega^2 = \frac{12d\hat{p}}{Eh^3}. \tag{47}$$

The differential equation of the shorter side wall is similar:

$$\frac{d^4\hat{u}}{dy^4} - \frac{12\rho_c}{Eh^2}\hat{u}\omega^2 = \frac{12d\hat{p}}{Eh^3}. \tag{48}$$

With the notation $K = \sqrt[4]{12\rho_c/(Eh^3)}$ the general solutions of these ODE-s are

$$\hat{w}(x) = Q\cosh\left(K\sqrt{\omega}x\right) + S\cos\left(K\sqrt{\omega}x\right) - \frac{12d\hat{p}}{Eh^3K^4\omega^2} \tag{49}$$

and

$$\hat{u}(y) = R\cosh\left(K\sqrt{\omega}y\right) + T\cos\left(K\sqrt{\omega}y\right) - \frac{12d\hat{p}}{Eh^3K^4\omega^2}. \tag{50}$$

The boundary conditions are identical with the previous ones. The solutions satisfying the boundary conditions using the side length ratio s again and denoting the constant term in the above differential equations by C result for the coefficients Q, S, R, T:

$$Q = -\frac{C}{\cosh\Omega}\frac{\tan\Omega + \tan(s\Omega)}{\tanh\Omega + \tan\Omega + \tanh(s\Omega) + \tan(s\Omega)} = -\frac{C}{\cosh\Omega}\mu, \tag{51}$$

$$R = -\frac{C}{\cosh(s\Omega)}\mu, \tag{52}$$

$$S = -\frac{C}{\cos\Omega}(1-\mu), \tag{53}$$

$$T = -\frac{C}{\cosh(s\Omega)}(1-\mu), \tag{54}$$

$$\tag{55}$$

with the notation $\Omega = K\sqrt{\omega}L/2$. Again the deflections of the four side walls can be integrated giving the change of area of channel cross section now depending on the excitation frequency ω. The final result for the function Φ is now

$$\Phi(\omega) = \frac{L^3}{15Eh^3s}\frac{45}{\Omega^5}\left(\frac{1}{\frac{1}{\tan\Omega+\tan(s\Omega)} + \frac{1}{\tanh\Omega+\tanh(s\Omega)}} - \Omega\frac{1+s}{2}\right) \tag{56}$$

and the wave velocity is

$$a = \frac{c}{\sqrt{1+\rho c^2\Phi(\omega)}}. \tag{57}$$

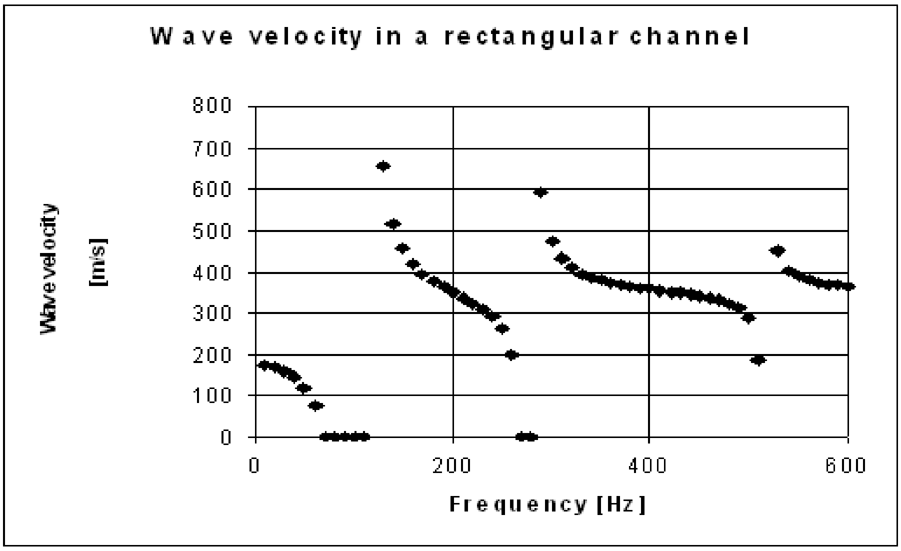


Figure 6: Wave velocity in a rectangular channel

4 Unsteady 1D slightly incompressible fluid flow

4.1 Governing equations

Let us start with the 1D incompressible equation of motion and continuity equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = S(x,t) \quad \text{where} \quad S(x,t) = -g\frac{dz}{dx} - \frac{\lambda}{2D}|v|v + a_x, \tag{58}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0 \tag{59}$$

Here $S(x,t)$ is the source term, including acceleration due to the inclination of the pipe ($g\,dz/dx$), friction and possible acceleration in the horizontal direction a_x.

Let us assume that the fluid is barotropic, i.e. $\rho = \rho(p)$. The continuity equation turns into

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = \underbrace{\frac{d\rho}{dp}}_{a^{-2}}\frac{\partial p}{\partial t} + \rho\frac{\partial v}{\partial x} + v\frac{d\rho}{dp}\frac{\partial p}{\partial x} = \frac{1}{a^2}\left(\frac{\partial p}{\partial t} + \rho a^2\frac{\partial v}{\partial x} + v\frac{\partial p}{\partial x}\right) = 0, \tag{60}$$

where a is the sonic velocity. Now we calculate a^2(60)+ρa (58):

$$\frac{\partial\left(p + \rho a v\right)}{\partial t} + (v+a)\frac{\partial\left(p + \rho a v\right)}{\partial x} = \rho a S(x,t). \tag{61}$$

Computing a^2(60)-ρa(58) gives

$$\frac{\partial\left(p - \rho a v\right)}{\partial t} + (v-a)\frac{\partial\left(p - \rho a v\right)}{\partial x} = -\rho a S(x,t). \tag{62}$$

Note that the above derivatives are directional derivates:

$$\frac{\mathcal{D}^+\clubsuit}{\mathcal{D}t} := \frac{\partial \clubsuit}{\partial t} + (v+a)\frac{\partial \clubsuit}{\partial x} \quad \text{is a derivative along the line} \frac{dx}{dt} = v + a$$

$$\frac{\mathcal{D}^-\clubsuit}{\mathcal{D}t} := \frac{\partial \clubsuit}{\partial t} + (v-a)\frac{\partial \clubsuit}{\partial x} \quad \text{is a derivative along the line} \frac{dx}{dt} = v - a$$

Hence, by defining $\alpha = p + \rho a v$ and $\beta = p - \rho a v$, the *ordinary differential equations* to be solved are

$$\frac{\mathcal{D}^+\alpha}{\mathcal{D}t} = \rho a S(x,t) \quad \text{and} \quad \frac{\mathcal{D}^-\beta}{\mathcal{D}t} = -\rho a S(x,t). \tag{63}$$

4.2 Application on pressurized liquid pipeline systems

In the case of pressurized liquid pipelines (e.g. water distribution systems or oil pipeline systems) the flow velocity in the pipeline v is typically in the range of a few m/s while the wave velocity a is in the range of $1000\,m/s$. The bulk modulus of water is $B = 2.1$GPa, which gives $a = \sqrt{B/\rho} = \sqrt{2.1 \times 10^3} \approx 1400\,m/s$. Hence the slope of te characetristic lines $v \pm a$ is hardly effected by the fluid velocity. The assumption $v \ll a$ allows the usage of a fix grid as the characteristic slopes are constant: $dx/dt = \pm a$, as shown in Figure 7.

4.2.1 Update of the internal points

A simple numerical scheme can be built onto (63), see Figure 7. We use a special grid that strictly satisfies

$$\frac{\Delta x}{\Delta t} = a, \tag{64}$$

meaning that if e.g. the spatial grid is set, the time step cannot be chosen arbitrarily but must be computed based on (64). The first step updates the α and β values in the internal points

$$\frac{\alpha_j^{i+1} - \alpha_{j-1}^i}{\Delta t} = \rho a S_{j-1}^i \quad \rightarrow \quad \alpha_j^{i+1} = \alpha_{j-1}^i + \Delta t \rho a S_{j-1}^i \quad j = 2 \dots N-1 \tag{65}$$

$$\frac{\beta_j^{i+1} - \beta_{j+1}^i}{\Delta t} = -\rho a S_{j+1}^i \quad \rightarrow \quad \beta_j^{i+1} = \beta_{j+1}^i - \Delta t \rho a S_{j+1}^i \quad j = 2 \dots N-1. \tag{66}$$

Then, we compute pressure and velocity simply by

$$p = \frac{\alpha + \beta}{2} \quad \text{and} \quad v = \frac{\alpha - \beta}{2\rho a}. \tag{67}$$

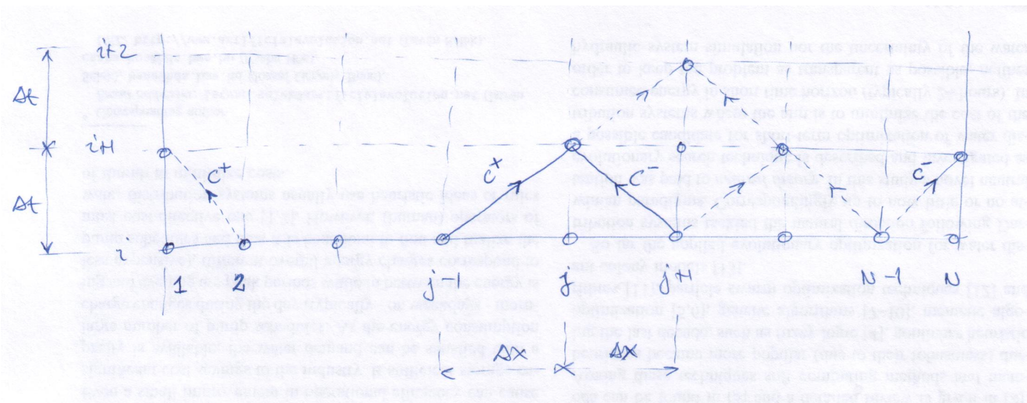


Figure 7: Incompressible MOC, numerical scheme.

4.2.2 Boundary conditions

At the boundary points we have only one characteristic equation. For example, on the left boundary $j = 1$ based on (66) we have

$$p_1^{i+1} - \rho a v_1^{i+1} = p_2^i \rho a v_2^i - \Delta t \rho a S_2^i := K_l \tag{68}$$

Another equation comes from the boundary condition:

Prescribed velocity: v_1^{i+1} is known, thus p_1^{i+1} can be computed based on (68).

Prescribed pressure: p_1^{i+1} is known, thus v_1^{i+1} can be computed based on (68).

Prescribed total pressure: $p_1^{i+1} + \frac{\rho}{2}\left(v_1^{i+1}\right)^2$ is known and must be solved together with (68).

Pump: the pump performance curve $H(Q)$ is known. We have $p_1^{i+1} = p_s + \rho g H(Q)$, where p_s is the pressure at the suction side of the pump. Furthermore, we have $Q = A_p v_1^{i+1}$ (A_p is the cross section area of the pipe), thus the equations to be solved for p_1^{i+1} and Q are

$$p_1^{i+1} = p_s + \rho g H(Q) \quad \text{and} \quad p_1^{i+1} - \rho a \frac{Q}{A_p} = K_l, \tag{69}$$

which can be rewritten as a single equation for Q:

$$p_s + \rho g H(Q) - \rho a \frac{Q}{A_p} = K_l. \tag{70}$$

TODO Add pump revolution number update here.

The performance curve of a pump $H(Q)$ depends on the revolution number. That is:

$$p_1^{i+1} = p_s + \rho g H(Q, n). \tag{71}$$

If n is given, (68) and (71) can be solved for p_1^{j+1} and v_1^{j+1} using e.g. Newton's technique.

For varying pump revolution number, we need two more pieces of information:

1. $H(Q,n)$=?
2. How to update $n^j \to n^{j+1}$?

1. Affinity laws

$$\frac{Q_1}{Q_2} = \frac{n_1}{n_2}, \quad \frac{H_1}{H_2} = \left(\frac{n_1}{n_2}\right)^2 \quad \Rightarrow \quad \frac{p_1}{p_2} = \left(\frac{n_1}{n_2}\right)^3 \tag{72}$$

Suppose we have

$$H_1(Q) = a_0 + a_1 Q_1 + a_2 Q_1^2 + a_3 Q_1^3 + \dots \tag{73}$$

at revolution number n_1. Then by substitution we get

$$\underbrace{H_2 \left(\frac{n_1}{n_2}\right)^2}_{H_1} = a_0 + a_1 \left(\frac{n}{n_2}\right) Q_2 + a_2 \left(\frac{n_1}{n_2}\right)^2 Q_2^2 + \dots \tag{74}$$

Thus $H(Q,n)$ can be computed in the following way:

$$H(Q,n) = a_0 \left(\frac{n}{n_1}\right)^2 + a_1 \left(\frac{n}{n_1}\right) Q + a_2 Q^2 + a_3 \left(\frac{n_1}{n}\right) Q^3 + \dots \tag{75}$$

where a_i are the fit coefficients of the performance curve at n_1.

2. Pump revolution number update

$$\Theta \cdot \varepsilon = M_{electr.} - M_{hydraulic} \tag{76}$$

where $M_{electr.}$ is a catalogue data (see Figure 8) and $M_{hydr} = \frac{P_{input}}{\omega}$ where $P_{input}(Q)$ is also a catalogue data at n_1 (see Figure 9).

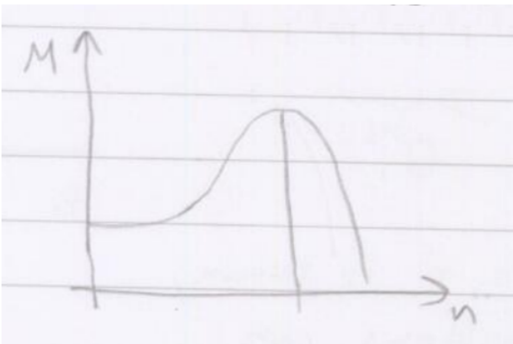


Figure 8: Torque of an electric motor at different revolution number (schematic figure).

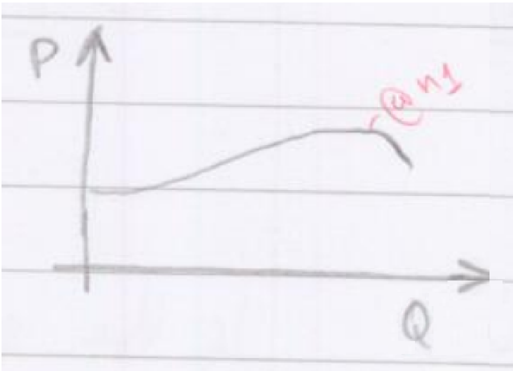


Figure 9: P-Q graph of an electric motor (schematic figure).

From

$$P(Q,n) = b_0 \left(\frac{n}{n_1}\right)^3 + b_1 \left(\frac{n}{n_1}\right)^2 Q + b_2 \left(\frac{n}{n_1}\right) Q^2 + \dots \tag{77}$$

and

$$\varepsilon = \frac{d\omega}{dt} = 2\pi \frac{dn}{dt} \tag{78}$$

we get:

$$\Theta \cdot 2\pi \frac{n^{j+1} - n^j}{\Delta t} = M_{electr.} - \frac{P_{in}(Q^j, n^j)}{2\pi n^j} \tag{79}$$

Summary of pump BC (for pump run-off) Input: $H(Q)$ and $P(Q)$ at n_1 reference revolution number from catalogue. After curve fitting we get coefficients $a_0, a_1, \dots$ and $b_0, b_1, \dots$.

In case of a pump run-off $M_{electr.} = 0$ (see Figure 10). In case of a pump run-up we need $M(n)$ (see Figure 9).

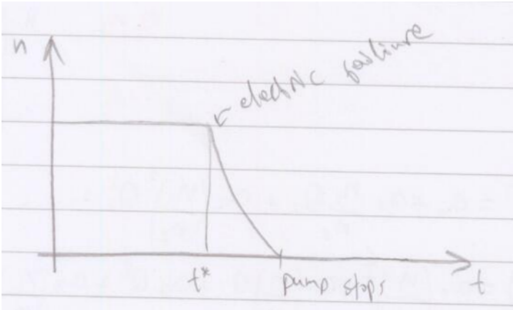


Figure 10: Change of revolution number in case of electric failure (schematic figure).

Update steps:

1. Update $n^j \to n^{j+1}$ with old Q^j value by (79) with $M_{electr.} = 0$
2. Solve (69) for p_1^{j+1} and v_1^{j+1} (knowing that $v_1^{j+1} = \frac{Q}{A_p}$).

5 Impedance method

The impedance technique presented in this chapter allows the calculation of hydraulic eigenfrequencies of pipeline systems. The basic frequency of a pipeline of length L and sonic velocity a is $f = a/(2L)$, where $2L/a$ is the time scale of the pipe; i.e. the time needed for a pressure wave to travel to the other end of the time and come back. However, for more complicated pipelines, it is not straightforward to cope with the interaction between different pipe segments.

The impedance technique assumes *periodic flow* in the pipeline and connects the amplitude of the excitation at one end with the response amplitude on the other end of the pipe, for arbitrary excitation frequency. Thus with the help of this technique, it is possible to construct resonance diagrams of complex (tree-like or looped) pipeline systems.

5.1 Basic theory

We start from the one dimensional continuity and momentum equation. The convective terms are neglected in both equations. We denote the sum of the static pressure p_{st} and hydrostatic pressure $\rho g h$ by p

$$p = p_{st} + \rho g h,$$

thus the basic equations are:

$$\frac{\partial p}{\partial x} + \rho\frac{\partial v}{\partial t} + \frac{\rho\lambda}{2d} v\,|v| = 0, \tag{80}$$

$$\frac{1}{a^2}\frac{\partial p}{\partial t} + \rho\frac{\partial v}{\partial x} = 0. \tag{81}$$

We consider *only periodic flows*. Let the mean values of pressure and velocity be $\bar{p}$ and $\bar{v}$ respectively, and the periodic parts be denoted by p and v:

$$p = \bar{p} + p'; \quad v = \bar{v} + v'. \tag{82}$$

The mean values are defined by the time integrals:

$$\bar{p}\,(x) = \frac{1}{T}\int_0^T p(t,x)dt \quad \text{and} \quad \bar{v}(x) = \frac{1}{T}\int_0^T v(t,x)dt. \tag{83}$$

In order to substitute (82) into (80) and (81) one has to differentiate the pressure and velocity:

$$\frac{\partial p}{\partial x} = \frac{\partial \bar{p}}{\partial x} + \frac{\partial p'}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial \bar{v}}{\partial x} + \frac{\partial v'}{\partial x}. \tag{84}$$

The derivatives with respect to time are similar. By Eq. (83)

$$\frac{\partial \bar{p}}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \bar{v}}{\partial t} = 0. \tag{85}$$

The mean values are solutions of Eqs. (80), (81) because they mean the unperturbed case:

$$\frac{\partial \bar{p}}{\partial x} = -\frac{\lambda\rho}{2d}\bar{v}\,|\bar{v}| = -\frac{\lambda\rho}{2d}\bar{v}^2 \tag{86}$$

and

$$\frac{\partial \bar{v}}{\partial x} = 0. \tag{87}$$

Further, we suppose that the velocity perturbation is much smaller than the mean value: $v' \ll \bar{v}$. Then we can neglect v'^2 in the turbulent friction term:

$$\frac{\lambda\rho}{2d}v^2 \cong \frac{\lambda\rho}{2d}\left(\bar{v}^2 + 2\bar{v}v'\right). \tag{88}$$

Substituting (84) – (88) into (80) and (81) we get:

$$\frac{\partial p'}{\partial x} + \rho\frac{\partial v'}{\partial t} + R_0 v' = 0, \tag{89}$$

$$\frac{\partial p'}{\partial t} + \rho\, a^2\frac{\partial v'}{\partial x} = 0, \qquad \text{with} \tag{90}$$

$$R_0 = \frac{\rho\lambda}{d}\bar{v}. \tag{91}$$

By differentiating one of Eqs. (89) and (90) with respect to time, the other one with respect to x and subtracting the second from the first results in equations which contain only p' or v'. ($R = R_0/\rho$):

$$a^2\frac{\partial}{\partial x}(89) - \frac{\partial}{\partial t}(90): \qquad a^2\frac{\partial^2 p'}{\partial x^2} - \frac{\partial^2 p'}{\partial t^2} - R\frac{\partial p'}{\partial t} = 0, \tag{92}$$

$$-\frac{1}{\rho}\left[\frac{\partial}{\partial t}(89) - \frac{\partial}{\partial x}(90)\right]: \qquad a^2\frac{\partial^2 v'}{\partial x^2} - \frac{\partial^2 v'}{\partial t^2} - R\frac{\partial v'}{\partial t} = 0. \tag{93}$$

The similar equations are both solved by Fourier's separation method. This will result in complex pressure and velocity perturbations: Naturally, only the real part has physical meaning. $\left(\imath = \sqrt{-1}\right)$

$$p'(x,t) = e^{\imath\omega t}\left(Ae^{\gamma x} + Be^{-\gamma x}\right). \tag{94}$$

Putting this into (90) and integrating over the pipe length x we get

$$v'(x,t) = \frac{\omega}{\imath\rho a^2\gamma}e^{\imath\omega t}\left(Ae^{\gamma x} - Be^{-\gamma x}\right). \tag{95}$$

In the above equations ω is the frequency of excitation, A and B will be determined by the boundary values, γ is called *propagation constant*. It is defined by

$$\gamma^2 = -\frac{\omega^2}{a^2} + \imath\frac{R\omega}{a^2}.$$

As a new parameter the hydraulic impedance is introduced, it is the ratio of pressure perturbation p' and velocity perturbation v':

$$Z(x) = \frac{p'}{v'} = \frac{\imath\rho a^2\gamma}{\omega}\cdot\frac{Ae^{\gamma x} + Be^{-\gamma x}}{Ae^{\gamma x} - Be^{-\gamma x}} \tag{96}$$

The first brake is called characteristic impedance Z_c:

$$Z_c = -\frac{\imath \rho a^2 \gamma}{\omega} \tag{97}$$

The hydraulic impedance $Z(x)$ depends only on the space coordinate. At the upstream end of the hydraulic element, $Z_u = Z(x = 0)$. At the downstream end $Z_d = Z(x = L)$.

$$Z_u = Z(0) = \frac{p'(0,t)}{v'(0,t)} \tag{98}$$

$$Z_d = Z(L) = \frac{p'(L,t)}{v'(L,t)} \tag{99}$$

The coefficients A and B can be expressed with the pressure and velocity perturbation at the upstream end of the hydraulic element. Putting $x = 0$ into (94) and (95) one has

$$p'(0,t) = e^{\imath \omega t}(A + B) = P_u, \tag{100}$$

$$v'(0,t) = -\frac{e^{\imath \omega t}}{Z_c}(A - B) = V_u. \tag{101}$$

These give for A and B

$$A = \frac{1}{2e^{\imath \omega t}}\left(P_u - Z_c V_u\right), \tag{102}$$

$$B = \frac{1}{2e^{\imath \omega t}}(P_u + Z_c V_u). \tag{103}$$

The perturbations over the length of the element are now:

$$p'(x,t) = P_u \cosh \gamma x - Z_c V_u \sinh \gamma x, \tag{104}$$

$$v'(x,t) = -\frac{P_u}{Z_c} \sinh \gamma x + V_u \cosh \gamma x. \tag{105}$$

Using these equations we find a relation between the upstream (P_u, V_u) and downstream values (P_d, V_d) of perturbations:

$$P_d = p'(L,t) = P_u \cosh \gamma L - Z_c V_u \sinh \gamma L, \tag{106}$$

$$V_d = v'(L,t) = -\frac{P_u}{Z_c} \sinh \gamma L + V_u \cosh \gamma L. \tag{107}$$

Using (96) – (99) and Eqs. (104), (105) the hydraulic impedance $Z(x)$ can be written as:

$$Z(x) = \frac{Z_u - Z_c \tanh \gamma x}{1 - \frac{Z_u}{Z_c} \tanh \gamma x}.$$

With vector notation and putting $x = L$ one has:

$$\begin{pmatrix} P_d \\ V_d \end{pmatrix} = \begin{pmatrix} \cosh \gamma L & -Z_c \sinh \gamma L \\ -\frac{\sinh \gamma L}{Z_c} & \cosh \gamma L \end{pmatrix} \begin{pmatrix} P_u \\ V_u \end{pmatrix}.$$

The matrix is called *impedance matrix*. The resulting impedance matrix of hydraulic elements connected in series is the product of the impedance matrices of the individual elements. The following expression connects the impedances at the upstream and downstream ends of the element:

$$Z_d = \frac{Z_u - Z_c \tanh \gamma L}{1 - \frac{Z_u}{Z_c} \tanh \gamma L}.$$

5.2 Boundary conditions

Some simple cases are studied where either the upstream or the downstream impedance (Z_u or Z_d) can be found easily. The pressure perturbation is zero if the *downstream pressure has a fixed constant value* (open end to the atmosphere or a liquid tank with constant surface)

$$Z_d = \frac{p'(L,t)}{v'(L,t)} = 0$$

Closed end of a pipe, the velocity perturbation is zero, thus

$$Z_d = \infty$$

Dividing or combining pipes (or other hydraulic elements) results in a boundary condition where the pressure perturbation is common for all connected elements and continuity is fulfilled. For elements k being connected ($k = 1, 2, \ldots, K$):

$$p_1'(L_i,t) = p_2'(L_2,t) = \ldots = p_k'(L_K,t) \tag{108}$$

Supposing constant liquid density in elements having cross sections A_k gives:

$$\sum_{k=1}^{K} A_k \left(\bar{v}_k + v_k'\right) = 0.$$

Continuity must be fulfilled for the mean velocities $\bar{v}$ too,

$$\sum_{k=1}^{K} A_k v_k' = 0. \tag{109}$$

From (108) and (109)

$$\sum_{k=1}^{K} \frac{A_k}{Z_k} = 0.$$

The fact that the velocity is proportional to the square root of pressure difference in turbulent flow is used to formulate the impedance of a *throttle valve* at the downstream end of a pipe:

$$\bar{v} + v' = \mu \sqrt{\frac{2}{\rho} \left(\Delta \bar{p} + \Delta p'\right)}.$$

If the pressure outside the valve is constant we can write $\Delta\bar{p} = \bar{p}$ and $\Delta p' = p'$ thus:

$$\bar{v} + v' = \mu\sqrt{\frac{2}{\rho}\left(\bar{p} + p'\right)}. \tag{110}$$

Considering that in many cases $p' \ll \bar{p}$ we can write

$$\bar{p} + p' \cong \bar{p}\left(1 + \frac{p'}{2\bar{p}}\right)^2,$$

further, for steady flow

$$\bar{v} = \mu\sqrt{\frac{2}{\rho}\bar{p}}.$$

After substituting these into (110) we get:

$$v' = \bar{v}\frac{p'}{2\bar{p}}.$$

Thus the downstream impedance is

$$Z_d = \frac{2\bar{p}}{\bar{v}}.$$

This is a real number meaning that there is no phase shift between the velocity and pressure variation in this point. As the last step we formulate the *excitation* as a boundary condition. Both velocity and pressure excitation can be handled in a similar manner. Let's see the velocity excitation. If the real velocity excitation is a sinusoidal vibration with angular frequency ω and amplitude A_0:

$$\Re\left(v'\right) = A_0 \sin\omega t$$

then the complex form of v' is:

$$v' = A_0 e^{\imath\left(\omega t - \frac{\pi}{2}\right)}.$$

TODO Add application example.

6 Unsteady 1D open-surface flow in prismatic channel

6.1 Introduction

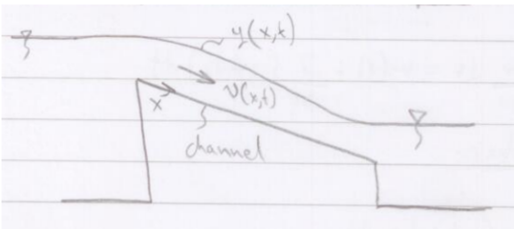


Figure 11: Open surface flow (schematic figure).

Typical applications:

- storm water system
- waste water

Water height changes and the flow-through area varies. **Friction loss:** $\frac{d}{dx}\left(\frac{\Delta p}{\rho g}\right) = \frac{v^2}{c^2 R_h}$ where $R_h = \frac{A}{p}$ the ration of the area and the perimeter (see Figure **??**) and $c = R_h^{1/6}\frac{1}{n}$ where n is the so-called *Manning's constant* (similar to λ).

6.2 Continuity equation and equation motion for open-surface flow

The usual form of continuity equation: $\frac{\partial \rho}{\delta t} + \frac{\partial}{\delta x}(\rho v) = 0$ is not suitable, because $\rho = const$ in our case; and the above form assumes constant flow-through area (which is not true for OCF).

6.2.1 Continuity equation

Using Taylor expansion in space around the inlet, the volume leaving at the end in dt time is (see Figure 12):

$$\rho_2 A_2 v_2 dt = \left(\rho_1 A_1 v_1 + \frac{\partial}{\partial x}(\rho A v)dx\right) dt \tag{111}$$

The mass of fluid in the stream tube is

$$m(t+dt) = m(t) + \frac{\partial m}{\partial t}dt = m(1) + \frac{\partial}{\partial t}(\rho A dx)dt \tag{112}$$

Putting these together we finally get

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A v) = 0 \tag{113}$$

Notice that

- if $A = const$ then get back the usual form: $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$ and
- if $\rho = const$ then $\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$ (the volumetric flow rate is $Q = Av$).

6.2.2 Equation of motion

From Newton's II. law we know that

$$\frac{dF}{dt} = F_{pressure} + F_{gravity} + F_{friction} \tag{114}$$

Using first order expansion

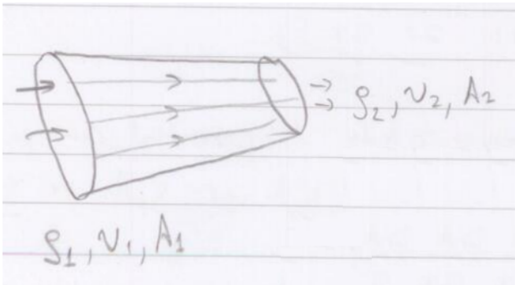


Figure 12: Open surface flow (schematic figure).

TODO

The general form is

$$\frac{\partial \rho v A}{\partial t} + \frac{\partial(v\rho v A)}{\partial x} = -A\left(\frac{\partial p}{\partial x} - \rho g S_0 + \rho g S_f\right) \tag{115}$$

where $S_0 = -dz/dx$ is the bed slope and $S_f = \frac{f}{2Dg}v^2$ is the friction loss.

6.2.3 General vector form for open-surface flow

For $\rho = 0$, the momentum equation can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial\, Qv}{\partial x} = -A\left(\frac{1}{\rho}\frac{\partial p}{\partial x} - gS_0 + gS_f\right). \tag{116}$$

Taking into account that $p = \rho g y$ and $-A\frac{1}{\rho}\frac{\partial p}{\partial x} = -\left(\frac{\partial A g y}{\partial x} - yg\frac{\partial A}{\partial x}\right)$, we have

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + Agy\right) = yg\frac{\partial A}{\partial x} + gA\left(S_0 - S_f\right). \tag{117}$$

6.3 The Saint-Venant equations

On typical way of writing the above equations is to introduce the free-surface width as $B(y) = \frac{\mathrm{d}A(y)}{\mathrm{d}y}$, hence, for example

$$\frac{\partial A}{\partial t} = \frac{dA}{dy}\frac{\partial y}{\partial t} := B(y)\frac{\partial y}{\partial t}. \tag{118}$$

The continuity equation can be written as

$$0 = \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = B\left(\frac{\partial y}{\partial t} + \frac{A}{B}\frac{\partial v}{\partial x} + v\frac{\partial y}{\partial x}\right) \tag{119}$$

The equation of motion is

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + gAy\right) = gyB\frac{\partial y}{\partial x} + Ag\left(S_0 - S_f\right) \tag{120}$$

$$A\frac{\partial v}{\partial t} + Bv\frac{\partial y}{\partial t} + A2v\frac{\partial v}{\partial x} + v^2B\frac{\partial y}{\partial x} + gA\frac{\partial y}{\partial x} + gyB\frac{\partial y}{\partial x} = gyB\frac{\partial y}{\partial x} + Ag\left(S_0 - S_f\right) \tag{121}$$

$$A\frac{\partial v}{\partial t} + Bv\left(\frac{\partial y}{\partial t} + \frac{A}{B}\frac{\partial v}{\partial x} + v\frac{\partial y}{\partial x}\right) + Av\frac{\partial v}{\partial x} + gA\frac{\partial y}{\partial x} + gyB\frac{\partial y}{\partial x} = gyB\frac{\partial y}{\partial x} + Ag\left(S_0 - S_f\right) \tag{122}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial y}{\partial x} = g\left(S_0 - S_f\right) \tag{123}$$

The famous Saint-Venant equations are

$$\frac{\partial y}{\partial t} + \frac{A}{B}\frac{\partial v}{\partial x} + v\frac{\partial y}{\partial x} + \frac{v}{B}\left.\frac{\partial A}{\partial x}\right|_{y=const.} = 0 \tag{124}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial y}{\partial x} = g\left(S_0 - S_f\right) \tag{125}$$

where $S_0 = -dz/dx$ and $S_f = \frac{\lambda}{2gd}v|v| = \frac{n^2}{R_h^{4/3}}v|v|$.

TODO: Explain friction modelling and the last term in the conti.eq.

6.4 Open-surface channel flow and gas dynamics

There are some similar phenomena in gas dynamics:

Gas dynamics	$\rightarrow$	**Open-surface flow**
$M = \frac{v}{a}$		$F_r = \frac{v}{c}$
M: Mach number		F_r: Froude number
$M < 1$ subsonic		$F_r < 1$ subcritical
$M > 1$ supersonic		$F_r > 1$ supercritical
shock wave		hydraulic jump

6.5 Method of characteristics for open-surface flow

6.5.1 MOC formulation for rectangular channel

The Saint-Venant equation can be re-written in terms of wave celerity $c = \sqrt{gy}$ instead of water depth. First, we compute

$$\frac{\partial y}{\partial t} = \frac{1}{g}\frac{\partial c^2}{\partial t} = \frac{2c}{g}\frac{\partial c}{\partial t} \quad \text{and} \quad \frac{\partial y}{\partial x} = \frac{1}{g}\frac{\partial c^2}{\partial x} = \frac{2c}{g}\frac{\partial c}{\partial x}. \tag{126}$$

we have now

$$\frac{2c}{g}\frac{\partial c}{\partial t} + \frac{A}{B}\frac{\partial v}{\partial x} + v\frac{2c}{g}\frac{\partial c}{\partial x} = 0 \tag{127}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + 2c\frac{\partial c}{\partial x} = g\left(S_0 - S_f\right) \tag{128}$$

Upon adding the two equations, i.e. EoM + $K \times$ Continuity:

$$\left(\frac{\partial v}{\partial t} + K\frac{2c}{g}\frac{\partial c}{\partial t}\right) + v\left(\frac{\partial v}{\partial x} + K\frac{2c}{g}\frac{\partial c}{\partial x}\right) + \left(2c\frac{\partial c}{\partial x} + Ky\frac{\partial v}{\partial x}\right) = 0. \tag{129}$$

Upon choosing $K = \pm\frac{g}{c}$, we have TODO: Explain the choice of K.

$$\left(\frac{\partial v}{\partial t} \pm 2\frac{\partial c}{\partial t}\right) + v\left(\frac{\partial v}{\partial x} \pm 2\frac{\partial c}{\partial x}\right) + c\left(2\frac{\partial c}{\partial x} \pm \frac{\partial v}{\partial x}\right) = g\left(S_0 - S_f\right) \quad \text{or, upon rearranging,} \tag{130}$$

$$\frac{\partial}{\partial t}\left(v \pm 2c\right) + \left(v \pm c\right)\frac{\partial}{\partial x}\left(v \pm 2c\right) = g\left(S_0 - S_f\right). \tag{131}$$

6.5.2 Numerical technique for the internal points

In the case of open-surface flows, there are several differences between the slightly compressible case (pressurized pipeline systems):

- $c = \sqrt{gy}$ not constant
- $v << c$ can happen (e.g:$y = 1\frac{m}{s} \rightarrow c \approx 3\frac{m}{s}$)
- even $v > c$ can happen!

Timestep selection

We have to choose appropriate timestep that meets the CFL criteria[1], thus we calculate the time needed to travel Δx distance with $v \pm c$ velocity, for each grid point: $\Delta t_i^+ = \frac{\Delta x}{|v_i + c_i|}$ and $\Delta t_i^- = \frac{\Delta x}{|v_i - c_i|}$ (see Figure 13). The final timestep for all grid points is than $\delta t = \min_i \left(\Delta t_i^+, \Delta t_i^-\right)$.

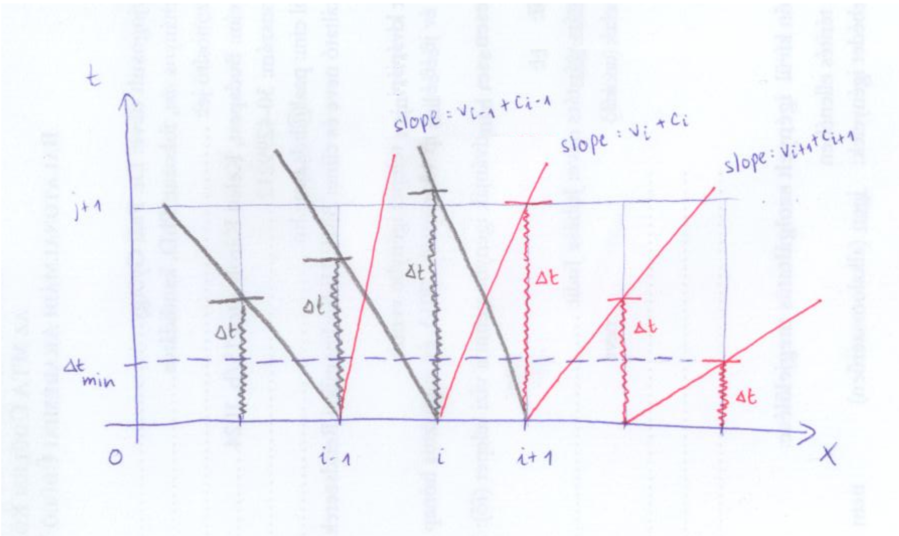


Figure 13: Updating internal points (schematic figure).

Internal points update

One has to find the location of that L(eft) and R(ight) point, from which the characteristic lines hit the required point on the new time level, see Figure 14. We use linear interpolation (see Figure 14), e.g. for the L point:

$$v_L(\delta x) = v_{i-1}\left(1 - \frac{\delta x}{\Delta x}\right) + v_i \frac{\delta x}{\Delta x} \tag{132}$$

$$c_L(\delta x) = c_{i-1}\left(1 - \frac{\delta x}{\Delta x}\right) + c_i \frac{\delta x}{\Delta x} \tag{133}$$

and the condition $\frac{\Delta x - \delta x}{\Delta t} = v_L(\delta x) + c_L(\delta x)$, which gives one single interpolation for δx. The R point location is found in a similar way.

Finally, we use the standard MOC method to update the water depth and the velocity: $\alpha_L = v_{i-1}^{j+1} + 2c_{i-1}^{j+1}$ and $\beta_L = v_{i+1}^{j+1} - 2c_{i+1}^{j+1}$.

Internal points update

Similar as for pressurized flow if $F_r < 1$, however, for $F_r > 1$ boundary condition (see Figures 15, 16 and 17). For supercritical inflow both y and v or c and v can be prescribed.

[1] TODO

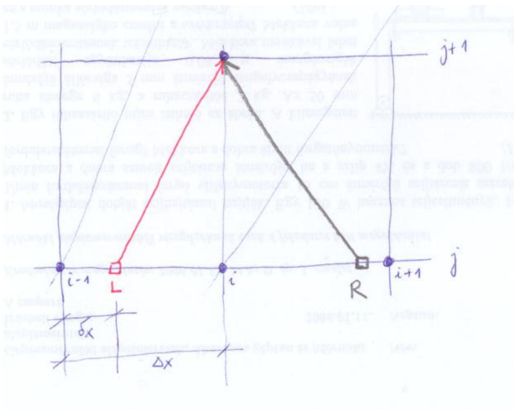


Figure 14: Linear interpolation flow (schematic figure).

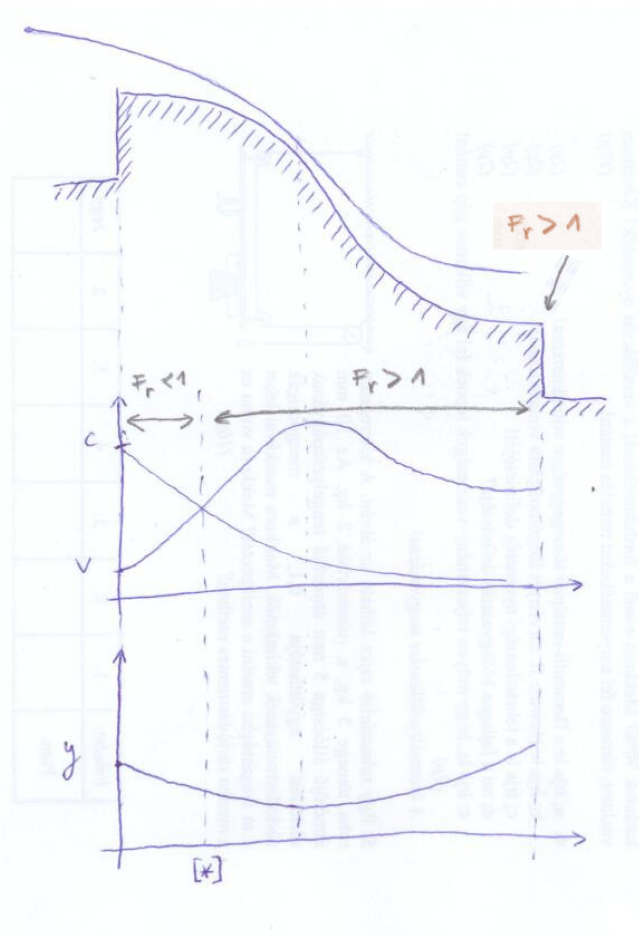


Figure 15: Example for supercritical boundary condition (schematic figure). Slopes: $v + c > 0$ and $v - c > 0$

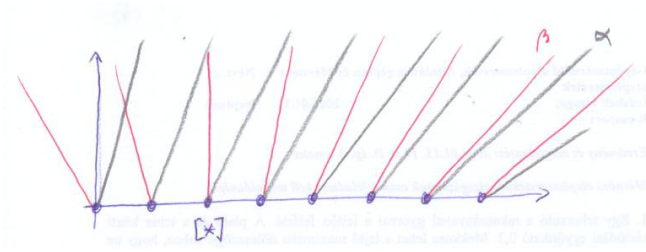


Figure 16: Change of clopes in supercritical case (schematic figure).

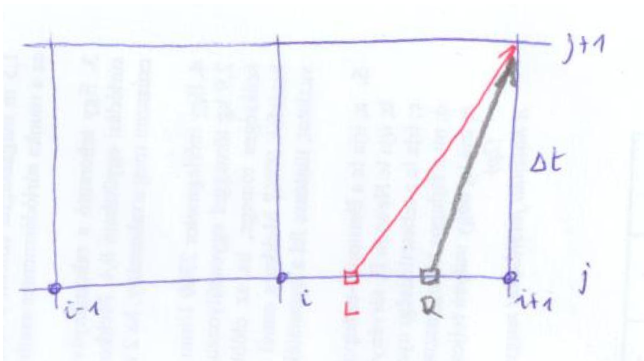


Figure 17: Linear interpolation in supercritical case (schematic figure). No outside information is needed.

7 Unsteady 1D compressible gas flow

7.1 Governing equations

Let us start off with the 1D integral form of the continuity equation

$$\frac{\partial}{\partial t}\int_V \rho dV + \oint_A \rho \underline{v}\, d\underline{A} = 0, \tag{134}$$

equation of motion

$$\frac{\partial}{\partial t}\int_V \rho \underline{v} dV + \oint_A \underline{v}\rho\underline{v}\, d\underline{A} = -\oint_A p\, d\underline{A} + \oint_A \underline{\underline{\tau}}\, d\underline{A}, \tag{135}$$

and energy equation

$$\frac{\partial}{\partial t}\int_V \rho e dV + \oint_A e\rho\underline{v}\, d\underline{A} = -\oint_A p\underline{v}\, d\underline{A} + \oint_A \underline{q}\, d\underline{A} \tag{136}$$

where $\underline{\underline{\tau}}$ is the stress tensor, $e = u + v^2/2$ (for an ideal gas, we have $du = c_V dT$) is the sum of internal energy and kinetic energy and $\underline{q}$ is the heat flux vector. We also need an equation of state of the form $p = f(\rho, u)$.

In what follows, we assume 1D flow, hence $\underline{v} = v(x,t)$.

Applying the divergence theorem on the continuity equation and expoiting that $V = A(x)x$, we have

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_V \frac{\partial}{\partial x}(\rho\underline{v})\, dV = \int \left(\frac{\partial\, \rho A}{\partial t} + \frac{\partial\, \rho v A}{\partial x}\right) dx = 0. \tag{137}$$

$$\frac{\partial\, \rho A}{\partial t} + \frac{\partial\, \rho v A}{\partial x} = 0. \tag{138}$$

The equation of motion takes the form

$$\frac{\partial}{\partial t}\int_V \rho v dV + \int_V \frac{\partial}{\partial x}\rho v^2\, dV = -\int \frac{\partial p}{\partial x} A dx + \oint_A \underline{\underline{\tau}}\, d\underline{A} \tag{139}$$

$$\int \left(\frac{\partial}{\partial t}\rho v A + \frac{\partial}{\partial x} A\rho v^2\right) dx = -\int \left(\frac{\partial (pA)}{\partial x} - p\frac{dA}{dx}\right) dx + \int F_s\, dx \tag{140}$$

$$\frac{\partial}{\partial t}\rho v A + \frac{\partial}{\partial x}\left(A\rho v^2 + pA\right) = p\frac{dA}{dx} + F_s = F_p + F_s. \tag{141}$$

The energy equation can be rewritten in a similar way. Finally, the system of equation to be solved are

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho v A) = 0 \tag{142}$$

$$\frac{\partial}{\partial t}(\rho v A) + \frac{\partial}{\partial x}\left(A\rho v^2 + pA\right) = F_p + F_s \tag{143}$$

$$\frac{\partial}{\partial t}(\rho e A) + \frac{\partial}{\partial x}(A\rho v e + p v A) = \dot{Q} \tag{144}$$

7.2 Isentropic MOC

Let us start off with some basic equations from thermodynamics. From the ideal gas law and the isentropic process, we have

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{1}{\kappa-1}} = \left(\frac{a}{a_0}\right)^{\frac{2}{\kappa-1}}, \tag{145}$$

$$\frac{\partial \rho}{\partial x} = \rho_0 \frac{2}{\kappa-1}\left(\frac{a}{a_0}\right)^{\frac{2}{\kappa-1}-1}\frac{\partial a}{\partial x}\frac{1}{a_0} = \rho_0 \frac{2}{\kappa-1}\frac{\rho}{\rho_0}\left(\frac{a}{a_0}\right)^{-1}\frac{\partial a}{\partial x}\frac{1}{a_0} = \frac{2}{\kappa-1}\frac{\rho}{a}\frac{\partial a}{\partial x}, \tag{146}$$

$$\frac{\partial \rho}{\partial t} = \frac{2}{\kappa-1}\frac{\rho}{a}\frac{\partial a}{\partial t} \quad \text{and} \tag{147}$$

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_{isentropic} \quad \to \quad \frac{\partial p}{\partial x} = a^2 \frac{\partial \rho}{\partial t}. \tag{148}$$

Now, the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = \frac{2}{\kappa-1}\frac{\rho}{a}\frac{\partial a}{\partial t} + \underbrace{v\frac{2}{\kappa-1}\frac{\rho}{a}\frac{\partial a}{\partial x}}_{v\frac{\partial \rho}{\partial x}} + \rho\frac{\partial v}{\partial x} \tag{149}$$

$$= \frac{2}{\kappa-1}\frac{\rho}{a}\left(\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x}\right) + \rho\frac{\partial v}{\partial x} = \frac{2}{\kappa-1}\frac{\rho}{a}\frac{\mathrm{d}a}{\mathrm{d}t} + \rho\frac{\partial v}{\partial x} = 0 \tag{150}$$

and the equation of motion becomes

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\mathrm{d}v}{\mathrm{d}t} + a\frac{2}{\kappa-1}\frac{\partial a}{\partial t} = 0 \tag{151}$$

Computing $\frac{a}{\rho}\frac{\kappa-1}{2}$ (150) $+\frac{\kappa-1}{2}$ (151) gives

$$0 = \frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\kappa-1}{2}a\frac{\partial v}{\partial x} + \frac{\kappa-1}{2}\frac{\mathrm{d}v}{\mathrm{d}t} + a\frac{\partial a}{\partial t} = \left(\frac{\mathrm{d}a}{\mathrm{d}t} + a\frac{\partial a}{\partial x}\right) + \frac{\kappa-1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + a\frac{\partial v}{\partial x}\right) \tag{152}$$

$$= \left(\frac{\partial a}{\partial t} + (a+v)\frac{\partial a}{\partial x}\right) + \frac{\kappa-1}{2}\left(\frac{\partial v}{\partial t} + (a+v)\frac{\partial v}{\partial x}\right) \tag{153}$$

$$:= \frac{\mathcal{D}^+ a}{\mathcal{D}^+ t} + \frac{\kappa-1}{2}\frac{\mathcal{D}^+ v}{\mathcal{D}^+ t} = \frac{\mathcal{D}^+}{\mathcal{D}^+ t}\left(a + \frac{\kappa-1}{2}v\right) := \frac{\mathcal{D}^+ \alpha}{\mathcal{D}^+ t} \tag{154}$$

A similar computation with $\frac{a}{\rho}\frac{\kappa-1}{2}$ (150) $-\frac{\kappa-1}{2}$ (151) gives

$$0 = \frac{\mathcal{D}^- \beta}{\mathcal{D}^- t} \quad \text{with} \quad \frac{\mathcal{D}^-}{\mathcal{D}^- t} = \frac{\partial}{\partial t} + (v-a)\frac{\partial}{\partial x} \quad \text{and} \quad \beta = a - \frac{\kappa-1}{2}v \tag{155}$$

7.3 Boundary conditions

In what follows we give all the equations for the beginning of the domain, i.e. the first node with subscript 1.

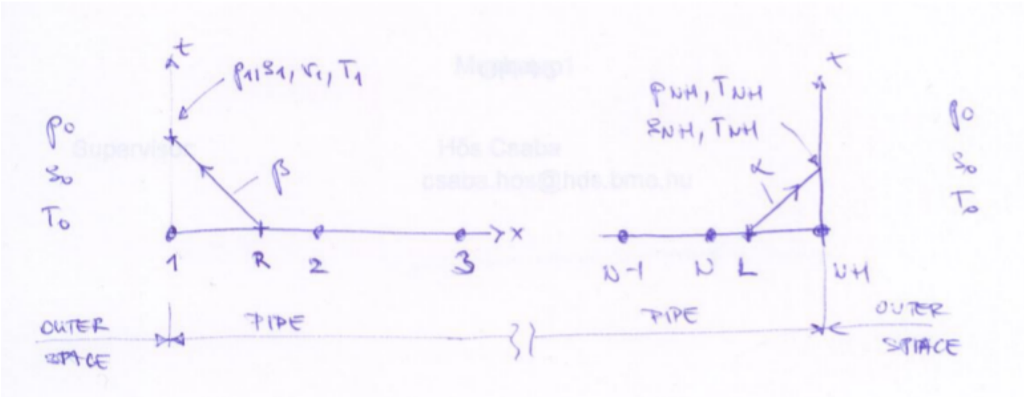


Figure 18: Boundary conditions in the case of subsonic flow.

7.3.1 Subsonic, isentropic inflow

The ambient gas properties (far away from the inlet) are p_0, T_0 and ρ_0. We are searching for T_1, a_1, v_1 and ρ_1. We have

- Characteristics from the inside of the pipe: $\beta_R = a_1 - \frac{\kappa-1}{2}v_1$ with β_R being known.
- Constant total temperature from the surroundings: $T_0 = T_1 + \frac{v_1^2}{2c_p} = \frac{a^2}{\kappa R} + \frac{v_1^2}{2c_p}$.

These two equations can be solved for a_1 and v_1.

7.3.2 Supersonic, isentropic inflow

The ambient gas properties (far away from the inlet) are p_0, T_0 and ρ_0. We are searching for T_1, a_1, v_1 and ρ_1. We have

- Characteristics from the inside of the pipe cannot be used as it is vertical.
- The velocity is $v_1 = a_1$.
- The inflow is still isentropic: $T_0 = T_1 + \frac{v_1^2}{2c_p} = \frac{a_1^2}{\kappa R} + \frac{a_1^2}{2c_p}$.

7.3.3 Subsonic, isentropic outflow

The ambient gas properties (far away from the inlet) are p_0, T_0 and ρ_0. We are searching for T_1, a_1, v_1 and ρ_1. We have

- Characteristics from the inside of the pipe: $\beta_R = a_1 - \frac{\kappa-1}{2} v_1$ with β_R being known.
- The pressure change is isentropic: $\frac{a_1}{a_R} = \left(\frac{p_1}{p_R}\right)^{\frac{\kappa-1}{2\kappa}}$.
- The pressure at the outlet is $p_0 = p_1$.

7.3.4 Supersonic, isentropic outflow

- We have $v_1 = -a_1$
- Characteristics from the inside of the pipe: $\beta_R = a_1 - \frac{\kappa-1}{2} v_1$ with β_R being known.
- We also have: $\alpha_L = a_1 + \frac{\kappa-1}{2} v_1$ with α_L being known.
- The inflow is still isentropic: $T_0 = T_1 + \frac{v_1^2}{2c_p} = \frac{a_1^2}{\kappa R} + \frac{a_1^2}{2c_p}$.

7.3.5 How do we decide if we have inflow or outflow?

At $x = 0$, for inflow, we have

$$2a_0 > 2a_1 = \alpha_1 + \beta_1 \quad \text{and} \tag{156}$$

$$\alpha_1 - \beta_1 = (\kappa - 1)\, v_1 > 0 \tag{157}$$

After adding these two equations, we obtain $\boxed{\beta_1 > a_0 \text{ as a condition for inflow}}$.

7.4 The Lax-Wendroff scheme

The governing equations can be written in a compact form

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = \mathcal{Q}, \tag{158}$$

with

$$\mathcal{U} = \begin{pmatrix} \rho A \\ \rho v A \\ \rho e A \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} \rho v A \\ \left(\rho v^2 + p\right) A \\ \left(\rho e v + p v\right) A \end{pmatrix}, \quad \text{and} \quad \mathcal{U} = \begin{pmatrix} 0 \\ F_p + F_s \\ \dot{Q} \end{pmatrix}. \tag{159}$$

Here the internal energy $e = c_V T$, $F_p = p\frac{\mathrm{d}A(x)}{\mathrm{d}x}$ and $F_s = A\frac{\rho}{2}\frac{\lambda}{D}v|v|$.

Note that if $\mathcal{U}$ is known, the primitive valiables can also be computed: $\rho = \mathcal{U}_1/A$, $v = \mathcal{U}_2/\mathcal{U}_1$ and $e = \mathcal{U}_3/\mathcal{U}_1$.

The LaxWendroff method, named after Peter Lax and Burton Wendroff, is a numerical method for the solution of hyperbolic partial differential equations, based on finite differences. It is second-order accurate in both space and time. The sceme is depicted in Figure 19 and consists of the following steps.

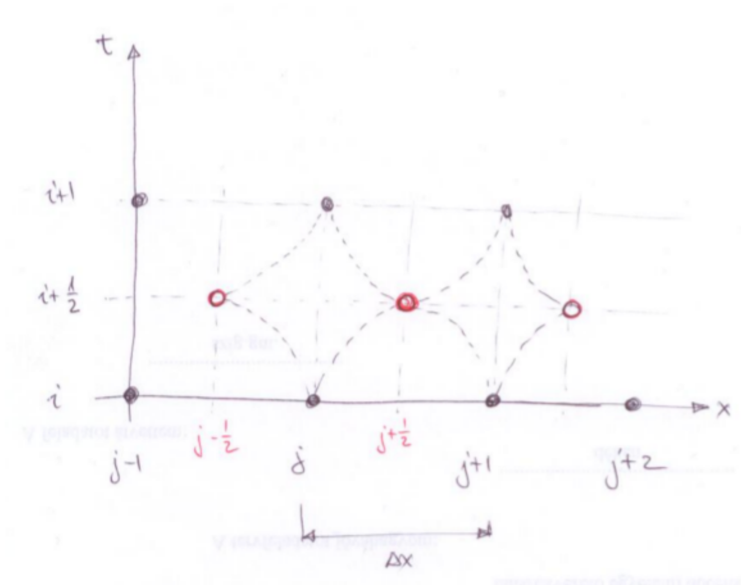


Figure 19: The Lax-Wendroff scheme.

1. Update $\mathcal{U}^{i+\frac{1}{2}}_{j+\frac{1}{2}}$ at the half time level and middle grid points from

$$\frac{\mathcal{U}^{i+\frac{1}{2}}_{j+\frac{1}{2}} - \mathcal{U}^{i}_{j+\frac{1}{2}}}{\Delta t/2} + \frac{\mathcal{F}^i_{j+1} - \mathcal{F}^i_j}{\Delta x} = \frac{\mathcal{Q}^i_{j+1} + \mathcal{Q}^i_j}{2}, \quad \text{where} \quad \mathcal{U}^i_{j+\frac{1}{2}} = \frac{\mathcal{U}^i_{j+1} + \mathcal{U}^i_j}{2}. \tag{160}$$

2. 'Unpack' the primitive valiables and compute $\mathcal{F}^{i+\frac{1}{2}}_{j+\frac{1}{2}}$.

3. Take a full time step to compute $\mathcal{U}^{i+1}_j$ with the help of $\mathcal{F}^{i+\frac{1}{2}}_{j+\frac{1}{2}}$:

$$\frac{\mathcal{U}^{i+1}_j - \mathcal{U}^i_j}{\Delta t} + \frac{\mathcal{F}^{i+\frac{1}{2}}_{j+\frac{1}{2}} - \mathcal{F}^{i+\frac{1}{2}}_{j-\frac{1}{2}}}{\Delta x} = \frac{\mathcal{Q}^{i+\frac{1}{2}}_{j+\frac{1}{2}} + \mathcal{Q}^{i+\frac{1}{2}}_{j-\frac{1}{2}}}{2}. \tag{161}$$

The time step Δt cannot be chosen arbitrarily, the timestep should be small enough to ensure that the scheme does not 'steps over' a cell with information propagation velocity $a + |v|$:

$$\Delta t_j < C \frac{\Delta x}{a_j + |v_j|}, \quad \Delta t = \min \Delta t_j, \tag{162}$$

where $C < 1$ is the 'safety' factor and $a_j = \sqrt{\kappa R T_j}$.

8 General framework on characteristic lines of PDEs

The equation of continuity – considering barotropic fluid

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial v}{\partial x} = 0, \tag{163}$$

and the equation of motion

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = S(x,t) \tag{164}$$

describe the 1D unsteady flow of a slightly compressible fluid [c1]where p is the pressure, v is the velocity, S is the friction and a is the wave velocity. We neglect the source term S. Let's find the solution of the above system of first order partial differential equations in the form of

[c1] *Viktor: added.*

$$p = \hat{p}e^{\imath(kx-\omega t)}, \quad v = \hat{v}e^{\imath(kx-\omega t)}. \tag{165}$$

Substituting the derivatives into the above equations and dropping the non-zero term $\imath e^{\imath(kx-\omega t)}$ we get

$$-\omega\hat{p} + kv\hat{p} + \rho a^2 k\hat{v} = 0, \quad \text{and} \quad -\omega\hat{v} + \frac{1}{\rho}k\hat{p} + kv\hat{v} = 0 \quad \text{respectively} \tag{166}$$

which is a linear homogeneous system of algebraic equations for the amplitudes $\hat{p}$, $\hat{v}$. Only the trivial solution ($\hat{p} = 0$, $\hat{v} = 0$) exists if the determinant of the coefficient matrix is non-zero. Thus the determinant must be zero:

$$\begin{vmatrix} -\omega + kv & \rho a^2 k \\ \frac{k}{\rho} & -\omega + kv \end{vmatrix} = 0, \tag{167}$$

that is $(-\omega + kv)^2 - a^2k^2 = 0$, or $-\omega + kv = \pm ak$, which after rearranging reads $\omega = k(v \pm a)$. Substituting this into $p(x,t)$ and $v(x,t)$, we get:

$$p(x,t) = \hat{p}e^{\imath k[x-(v\pm a)t]}, \quad v(x,t) = \hat{v}e^{\imath k[x-(v\pm a)t]}. \tag{168}$$

Thus if the source term S is zero the value of p and v does not change along the lines:

$$x - (v \pm a)t = \text{const.}, \tag{169}$$

these are the so-called *characteristics*. Physically this means that the shape of the pressure or velocity distribution along the pipe at some time t is kept but it is shifted to some other location determined by (169). This is typical for the propagation of wave forms.

8.1 Solution of general partial differential equations describing fluid flow

After having seen that the system of flow equations

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial v}{\partial x} = 0 \quad \text{and} \tag{170}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = S(x,t) \tag{171}$$

describe the propagation of pressure (and fluid speed) waves we try to formulate the PDE of a pressure wave. Differentiating (170) with respect to t:

$$\frac{\partial^2 p}{\partial t^2} + \frac{\partial v}{\partial t}\frac{\partial p}{\partial x} + v\frac{\partial^2 p}{\partial x \partial t} + \rho a^2 \frac{\partial^2 v}{\partial x \partial t} = 0. \tag{172}$$

Differentiating (171) with respect to x:

$$\frac{\partial^2 v}{\partial t \partial x} + \left(\frac{\partial v}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} = \frac{\partial S}{\partial x}. \tag{173}$$

After substitution

$$\frac{\partial^2 p}{\partial t^2} + \frac{\partial v}{\partial t}\frac{\partial p}{\partial x} + v\frac{\partial^2 p}{\partial x \partial t} + \rho a^2 \left(-v\frac{\partial^2 v}{\partial x^2} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2}\right) = \rho a^2 \left(\frac{\partial v}{\partial x}\right)^2 - \rho a^2 \frac{\partial S}{\partial x}. \tag{174}$$

Besides the derivatives of p the second derivative of the term $(v^2/2)$ occurs too. This can be eliminated by differentiating (170) with respect to x:

$$\rho a^2 \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 p}{\partial t \partial x} - \frac{\partial v}{\partial x}\frac{\partial p}{\partial x} - v\frac{\partial^2 p}{\partial x^2}. \tag{175}$$

Finally substituting this gives

$$\frac{\partial^2 p}{\partial t^2} + v\frac{\partial^2 p}{\partial x \partial t} - a^2\frac{\partial^2 p}{\partial x^2} + v\frac{\partial^2 p}{\partial x \partial t} + v^2\frac{\partial^2 p}{\partial x^2} = -\rho a^2 \frac{\partial S}{\partial x} - \frac{\partial v}{\partial t}\frac{\partial p}{\partial x} + \rho a^2 \left(\frac{\partial v}{\partial x}\right)^2 - v\frac{\partial v}{\partial x}\frac{\partial p}{\partial x}. \tag{176}$$

There are second derivatives of the pressure on the left hand side while all terms on the r.h.s. are of minor order. Thus the structure of our equation is:

$$(v^2 - a^2)\frac{\partial^2 p}{\partial x^2} + 2v\frac{\partial^2 p}{\partial x \partial t} + \frac{\partial^2 p}{\partial t^2} = F\left(t, x, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}, \frac{\partial p}{\partial x}\right). \tag{177}$$

Supposing that the velocity function $v(x,t)$ is known, one has a *PDE of* 2^{nd} *order* which is *linear in its higher order terms* for the function $p(x,t)$.

8.2 Classification of second order linear PDEs with constant coefficients. Canonical form and characteristic differential equation of 2nd order PDE-s.

Following notations are introduced:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad u = u(\mathbf{x}); \quad \mathbf{u}' = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \end{pmatrix}. \tag{178}$$

The second order differential operator L has the form:

$$Lu = \sum_{i=1}^{2}\sum_{k=1}^{2} a_{ik}(\mathbf{x})\frac{\partial^2 u}{\partial x_i \partial x_k} + f(\mathbf{x}, u, \mathbf{u}') =$$

$$a_{11}(\mathbf{x})\frac{\partial^2 u}{\partial x_1^2} + a_{12}(\mathbf{x})\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{21}(\mathbf{x})\frac{\partial^2 u}{\partial x_2 \partial x_1} + a_{22}(\mathbf{x})\frac{\partial^2 u}{\partial x_2^2} + f(\mathbf{x}, u, \mathbf{u}'). \tag{179}$$

The coefficients are continuous and symmetric that is $a_{ik}(x) = a_{ki}(x)$ thus

$$Lu = a_{11}(\mathbf{x})\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}(\mathbf{x})\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}(\mathbf{x})\frac{\partial^2 u}{\partial x_2^2} + f\left(\mathbf{x}, u, \mathbf{u}'\right). \tag{180}$$

The first part of this operator containing second order derivatives is called the main part. L is a linear operator since it is linear in the unknown function u.

Definition. *The differential operator L in point $\boldsymbol{x}$ is* elliptic, hyperbolic *or* parabolic *if the matrix $\boldsymbol{A}(\boldsymbol{x})$ composed by the coefficients $a_{ik}(\boldsymbol{x})$ in point $\boldsymbol{x}$ is* definite, indefinite *or* semi-definite.

The 2x2 matrix $\mathbf{A}$ in any point $\mathbf{x}$ has two eigenvalues: λ_1 and λ_2. We distinguish three different cases:

- $\lambda_{1,2} > 0$ or $\lambda_{1,2} < 0 \Longrightarrow L$ is *elliptic*
- $\lambda_1 > 0$ and $\lambda_2 < 0$ ($\mathbf{A}$ is indefinite) $\Longrightarrow L$ is *hyperbolic*
- $\lambda_1 > 0$ or $\lambda_1 < 0$ and $\lambda_2 = 0 \Longrightarrow L$ is *parabolic*

The PDE itself is called elliptic, hyperbolic or parabolic if L is elliptic, hyperbolic or parabolic. After this introduction we consider the second order linear PDE (180). Its second order main part is – with some different notation:

$$Lu = a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2}\dots \tag{181}$$

In order to determine the type of the PDE we must find the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}. \tag{182}$$

From

$$\det(\mathbf{A} - \lambda\mathbf{E}) = \begin{vmatrix} a(x,y) - \lambda & b(x,y) \\ b(x,y) & c(x,y) - \lambda \end{vmatrix} =$$

$$= \lambda^2 - [a(x,y) + c(x,y)]\lambda + a(x,y)c(x,y) - b^2(x,y) = \lambda^2 - \lambda\,\mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = 0. \tag{183}$$

We can now find the eigenvalues by solving this quadratic algebraic equation. We omit the arguments of the coefficients:

$$\lambda_{1,2} = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})}}{2} = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{(a+c)^2 - 4(ac - b^2)}}{2} =$$

$$= \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{(a-c)^2 + 4b^2}}{2}. \tag{184}$$

We see that the eigenvalues are always real as under the square root the argument is always positive; this is the corollary of $\mathbf{A}$ being symmetric. We denote the determinant by $d(x,y) = a(x,y)\,c(x,y) - b^2(x,y)$. The equation (183) has two roots, λ_1 and λ_2, thus the equation can be written in the form

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 0.$$

By comparing this with (183) we have

$$\lambda_1 + \lambda_2 = \mathrm{tr}(\mathbf{A}); \quad \lambda_1\lambda_2 = \det(\mathbf{A}) = d(x,y).$$

The 3 different cases are:

- $d(x,y) = 0 \Longrightarrow \lambda_1 = 0$ and $\lambda_2 = \mathrm{tr}(\mathbf{A})$, thus the PDE is *parabolic*
- $d(x,y) > 0 \Longrightarrow$ both eigenvalues are either positive or negative, thus the PDE is *elliptic*
- $d(x,y) < 0 \Longrightarrow \lambda_1 > 0$ and $\lambda_2 < 0$, thus the PDE is *hyperbolic*

From now on we shall only consider the hyperbolic case. In a one dimensional pipe flow the 2^{nd} order PDE for the pressure is

$$(v^2 - a^2)\frac{\partial^2 p}{\partial x^2} + 2v\frac{\partial^2 p}{\partial x \partial t} + \frac{\partial^2 p}{\partial t^2} = F\left(t, x, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}, \frac{\partial p}{\partial x}\right). \tag{185}$$

Here the former space coordinate y is denoted with t, that stands for time. Comparing this equation with the differential operator L we have

$$a(x,t) = v^2 - a^2; \quad b(x,t) = v; \quad c(x,t) = 1,$$

thus $\det(\mathbf{A}) = d(x,t) = v^2 - a^2 - v^2 = -a^2 < 0$; L is *hyperbolic.*

$$\mathrm{tr}(\mathbf{A}) = v^2 - a^2 + 1, \quad \lambda_{1,2} = \frac{v^2 - a^2 + 1 \pm \sqrt{(v^2 - a^2 - 1)^2 + 4v^2}}{2}.$$

8.3 Coordinate transformation resulting in a simpler form of operator L

Next we transform the independent variables x, t trough the functions $\xi = \xi(x,t)$, $\eta = \eta(x,t)$ resulting in a more simple structure of the left hand side of (185). Let the transformation functions be continuously differentiable. By the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\xi_x + \frac{\partial}{\partial \eta}\eta_x; \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\xi_t + \frac{\partial}{\partial \eta}\eta_t,$$

further

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial^2}{\partial \xi^2}\xi_x + \frac{\partial^2}{\partial \xi \partial \eta}\eta_x\right)\xi_x + \frac{\partial}{\partial \xi}\xi_{xx} + \left(\frac{\partial^2}{\partial \eta \partial \xi}\xi_x + \frac{\partial^2}{\partial \eta^2}\eta_x\right)\eta_x + \frac{\partial}{\partial \eta}\eta_{xx} =$$

$$= \frac{\partial^2}{\partial \xi^2}\xi_x^2 + 2\frac{\partial^2}{\partial \xi \partial \eta}\xi_x\eta_x + \frac{\partial^2}{\partial \eta^2}\eta_x^2 + \xi_{xx}\frac{\partial}{\partial \xi} + \eta_{xx}\frac{\partial}{\partial \eta}.$$

Similarly

$$\frac{\partial^2}{\partial x \partial t} = \frac{\partial^2}{\partial \xi^2}\xi_x\xi_t + \frac{\partial^2}{\partial \xi \partial \eta}(\xi_x\eta_t + \eta_x\xi_t) + \frac{\partial^2}{\partial \eta^2}\eta_x\eta_t + \ldots,$$

and

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \xi^2}\xi_t^2 + 2\frac{\partial^2}{\partial \xi \partial \eta}\xi_t\eta_t + \frac{\partial^2}{\partial \eta^2}\eta_t^2 + \ldots$$

Substituting all these derivatives into the l.h.s. of (185) [c1] (without p) we have

[c1] *Viktor: added.*

$$\left((v^2-a^2)\xi_x^2 + 2v\xi_x\xi_t + \xi_t^2\right)\frac{\partial^2}{\partial \xi^2} + 2\left((v^2-a^2)\xi_x\eta_x + v\left(\xi_x\eta_t + \eta_x\xi_t\right) + \xi_t\eta_t\right)\frac{\partial^2}{\partial \xi \partial \eta} + $$
$$+ \left((v^2-a^2)\eta_x^2 + 2v\eta_x\eta_t + \eta_t^2\right)\frac{\partial^2}{\partial \eta^2} = \ldots. \quad (186)$$

The first and third term have the same structure with the only difference that the first term contains the derivatives of ξ, while the third term is related to η. The above expression will be simpler if the 1^{st} and 3^{rd} term is set to zero:

$$(v^2-a^2)\xi_x^2 + 2v\xi_x\xi_t + \xi_t^2 = 0, \quad (v^2-a^2)\eta_x^2 + 2v\eta_x\eta_t + \eta_t^2 = 0.$$

From the first equation

$$\xi_t = \frac{-2v\xi_x \pm \sqrt{4v^2\xi_x^2 - 4(v^2-a^2)\xi_x^2}}{2} = (-v \pm a)\xi_x = -(v \pm a)\xi_x.$$

This is the equation of a $\xi(x,t) = const.$ line as the total difference of ξ is zero along such a line:

$$d\xi = \xi_x dx + \xi_t dt = 0, \quad \text{that is} \quad \xi_t = -\frac{dx}{dt}\xi_x.$$

By comparing the two formulae for ξ_t one has

$$\frac{dx}{dt} = v \pm a \quad (187)$$

and $\xi(x,t)$ is constant on the pair of lines defined by (187). Because of the formal similarity the function $\eta(x,t)$ is constant too. Equation (187) is called the characteristic differential equation of a PDE of 2^{nd} order which is linear in its higher order terms. In the present case the solutions of the two characteristic differential equations are two real characteristics:

$$x = (v-a)t + constant,$$
$$x = (v+a)t + constant,$$

that is

$$\xi = x - (v-a)t$$

and

$$\eta = x - (v + a)t.$$

The members of the two families of characteristics are $\xi = const.$ and $\eta = const.$ lines. The $\xi = const.$ and $\eta = const.$ lines span a net on the total x,t plane. An appropriate pair of values ξ,η can be attached to each point of this plane. This attachment is only unique if the different ξ lines do not intersect each other and the same holds for the different η lines. As the wave velocity a e.g. for acoustic waves is [c2] *Viktor Text added.* $a = \sqrt{\kappa RT} = \sqrt{\kappa p/\rho}$, thus if the pressure is changing during ~~thee~~ [c2]the propagation of the pressure wave then the wave velocity may change too. For open surface flows the wave may get steeper as the wave velocity is higher at the top of a wave than at its bottom. This leads finally to vertical wave fronts or to water jumps. In gas dynamics the analogous phenomenon is called shock wave. These discontinuities are excluded from our further investigations. After having found the transformation resulting in a simpler form of the main part of the differential equation we shall calculate this new form of the main part. The coefficient of the derivative $\frac{\partial^2 p}{\partial\xi\partial\eta}$ in Eq. (186) is $-4a^2$, thus the main part is

$$-4a^2 \frac{\partial^2 p}{\partial\xi\partial\eta} = \ldots.$$

This is the canonical form of the PDE, the type of the PDE is *hyperbolic*. The information on the type of the PDE is important as this will decide the kind of *initial and boundary conditions* which may be prescribed in order to solve the PDE. If for example the pressure is prescribed along some section of the x axis for $t = 0$: $p(x, t = 0)$ then inside the domain bordered by one $\eta = constant$ line starting from the left end of the section and by one $\xi = constant$ line starting from the right end of the section the pressure distribution $p(x,t)$ can be computed. Events occurring outside of this

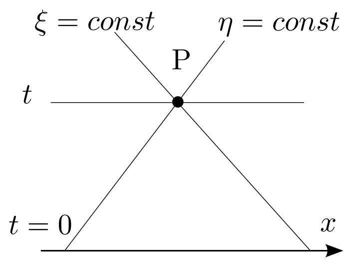


Figure 20: The information cone of point P

domain do not influence the function values at point P, the intersection of a ξ- and an η-line. Thus this triangular domain is called the information cone of point P (see Fig. 20). An observer situated at P will be informed within the time t only on the section of the x axis lying in the information cone. To know the flow trough the entire pipe length $0 \leq x \leq L$ the initial values at $t = 0$ and the boundary values at $x = 0$ and at $x = L$ must be known:

$$p(x = 0, t) \quad \text{and} \quad p(x = L, t).$$

Naturally the boundary values of $v(x, t)$ or some relation between pressure and velocity $p(v)$ is appropriate too.

9 Steady 2D supersonic gas flow (in the diffuser part of a Laval nozzle)

The flow is completely described by the Euler equation and the continuity equation.

$$1^{st} \text{ Euler:} \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x} \quad |\cdot\left(-\tfrac{u}{a^2}\right) \tag{188}$$

$$2^{nd} \text{ Euler:} \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial y} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial y} \quad |\cdot\left(-\tfrac{v}{a^2}\right) \tag{189}$$

By multiplying with the given terms and adding these equations we get

$$-\frac{u^2}{a^2}\frac{\partial u}{\partial x} - \frac{uv}{a^2}\frac{\partial u}{\partial y} - \frac{uv}{a^2}\frac{\partial v}{\partial x} - \frac{v^2}{a^2}\frac{\partial v}{\partial y} = \frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{v}{\rho}\frac{\partial \rho}{\partial y}. \tag{190}$$

Continuity:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 = \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} \quad |: \rho \tag{191}$$

If we divide by ρ we get the right hand side sum of Eq. (190): $\frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{v}{\rho}\frac{\partial \rho}{\partial y} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$. Substituting this into Eq. (190) the equation of motion has been derived for the supersonic flow:

$$-\frac{u^2}{a^2}\partial u \partial x - \frac{uv}{a^2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{v^2}{a^2}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} =$$
$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} + \left(1 - \frac{v^2}{a^2}\right)\frac{\partial v}{\partial y} - \frac{uv}{a^2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0. \tag{192}$$

Assuming isentropic flow the flow is rotation free too. There exists a velocity potential $\Phi(x, y)$. The velocity components u and v can be derived as derivatives of the potential: $u = \frac{\partial \Phi}{\partial x} = \Phi_x$, $v = \frac{\partial \Phi}{\partial y} = \Phi_y$. We put these and the second order derivatives into Eq. (192):

$$\left(1 - \frac{\Phi_x^2}{a^2}\right)\Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right)\Phi_{yy} - 2\frac{\Phi_x\Phi_y}{a^2}\Phi_{yx} = 0. \tag{193}$$

One can prove that this is a 2^{nd} order PDE of hyperbolic type. We introduce new independent variables ξ, η instead of x and y. Before we do it we must find an equation for the sonic velocity a too. In steady isentropic flow the energy equation is $h_{total} = constant$. The total enthalpy is:

$$h_{total} = h + \frac{w^2}{2} = c_pT + \frac{w^2}{2} = \frac{\kappa RT}{\kappa - 1} + \frac{w^2}{2} =$$
$$\frac{a^2}{\kappa - 1} + \frac{w^2}{2} = \frac{1}{\kappa - 1}\left(a^2 + w^2\frac{\kappa - 1}{2}\right) = constant \tag{194}$$

The Laval nozzle takes the air from the free atmosphere where the fluid velocity w is zero and the sonic speed a_0 is known. Thus $a^2 + w^2\frac{\kappa-1}{2} = a_0^2$ or by rearranging

$$a^2 = a_0^2 - w^2\frac{\kappa - 1}{2}. \tag{195}$$

Now the transformation of Eq. (193) must be done. In the same way as in Chapter **??** we differentiate Φ with respect to $x(\xi,\eta)$ and $y(\xi,\eta)$ as many times as needed.

$$\left(\left(1-\frac{u^2}{a^2}\right)\xi_x^2 + \left(1-\frac{v^2}{a^2}\right)\xi_y^2 - \frac{2uv}{a^2}\xi_x\xi_y\right)\frac{\partial^2\Phi}{\partial\xi^2} + (\dots)\frac{\partial^2\Phi}{\partial\xi\partial\eta} + (\dots)\frac{\partial^2\Phi}{\partial\eta^2} + \dots = 0. \tag{196}$$

In order to simplify this equation the first and third bracketed term must be zero, they have identical structure. The first term to be made zero us a quadratic equation for ξ_x. Solving for this we have:

$$\xi_x = \frac{\frac{2uv}{a^2} \pm \sqrt{\frac{4u^2v^2}{a^4}\xi_y^2 - 4\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{u^2}{a^2}\right)\xi_y^2}}{2\left(1-\frac{u^2}{a^2}\right)} = \frac{\frac{uv}{a^2} \pm \sqrt{\frac{u^2+v^2}{a^2} - 1}}{1-\frac{u^2}{a^2}}\xi_y. \tag{197}$$

On the other hand if we consider the $\xi = constant$ line of the new coordinate system, then its total differential is zero: $d\xi = \xi_x dx + \xi_y dy = 0$ which means that $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$. From Eq. (197) the tangent of the $\xi = constant$ characteristic line is:

$$\frac{dy}{dx} = -\frac{\frac{uv}{a^2} \pm \sqrt{\frac{u^2+v^2}{a^2} - 1}}{1-\frac{u^2}{a^2}}. \tag{198}$$

One sign is for $\xi = constant$, the other one for $\eta = constant$. Instead of the velocity components u and v we can introduce the components of the velocity vector w ($u = w\cos\vartheta$, $v = w\sin\vartheta$), further the Mach number: $M^2 = \frac{w^2}{a^2} = \frac{u^2+v^2}{a^2}$. Then the tangents read

$$\frac{dy}{dx} = -\frac{M^2\sin\vartheta\cos\vartheta \pm \sqrt{M^2-1}}{1-M^2\cos^2\vartheta}.$$

If we perturb a supersonic flow of velocity w at some point than this perturbation propagates in the inside of the Mach cone (see Fig. 21). In 1 second the perturbed gas spot moves to a distance of w meter and the perturbation is spreading inside a circle of radius a meter. From these the sinus of the Mach cone angle α is: $\sin\alpha = \frac{a}{w} = \frac{1}{M}$. We can write – without going into trigonometric details – that

$$\frac{dy}{dx} = \tan(\vartheta + \alpha) \quad \text{for} \quad \xi = constant \text{ lines and} \tag{199}$$

$$\frac{dy}{dx} = \tan(\vartheta - \alpha) \quad \text{for} \quad \eta = constant \text{ lines.} \tag{200}$$

Now the flow angle ϑ can be introduced also into Eq.(192). Again without details we get

$$\frac{\partial\vartheta}{\partial\xi} - \frac{\sqrt{M^2-1}}{w}\frac{\partial w}{\partial\xi} = 0 \quad \text{for} \quad \xi = constant \text{ lines and} \tag{201}$$

$$\frac{\partial\vartheta}{\partial\xi} + \frac{\sqrt{M^2-1}}{w}\frac{\partial w}{\partial\xi} = 0 \quad \text{for} \quad \eta = constant \text{ lines.} \tag{202}$$

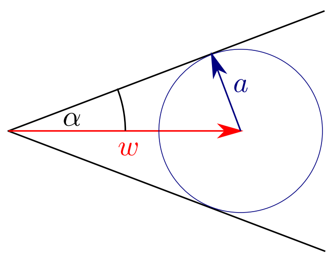


Figure 21: Mach cone

The computation must start just downstream of the throat of the Laval nozzle where the M number is slightly above $M = 1$. The computation may proceed inside a domain bordered by the starting vertical and two characteristic lines. Further downstream the symmetry boundary condition at the horizontal axis and the solid boundary with prescribed flow angle ϑ must be considered. The sonic speed must be recalculated in all newly computed points from Eq. (195). From the sonic velocity the absolute temperature $T = a^2/(\kappa R)$ can be calculated. As we have assumed isentropic flow $\left(T/p^{\frac{\kappa-1}{\kappa}} = constant\right)$ is valid thus we find the pressure distribution too and finally by the ideal gas law the density $\rho = p/(RT)$ can be computed. The contour of a Laval nozzle downstream of the throat can be easily defined by some simple formula as e.g. $y = a - b \cdot \cos(x^c)$ (see Fig. 22). The parameters must be adjusted appropriately.

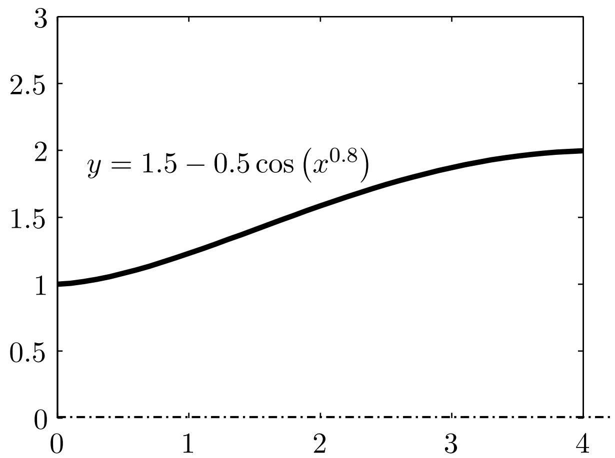


Figure 22: Contour of a Laval nozzle