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SCRIPT

Theoretical Acoustics

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Contents

1. Fluid Dynamics

We consider the motion of fluids in the continuum approximation, so that a body $\mathcal{B}$ is composed of particles $\mathcal{R}$ as displayed in Fig. 1.1. Thereby, a particle $\mathcal{R}$ already represents a macroscopic element. On the one hand a particle has to be small enough to describe the deformation accurately and on the other hand large enough to satisfy the assumptions of continuum theory. This means that the physical quantities density ρ, pressure p, velocity $\boldsymbol{v}$,

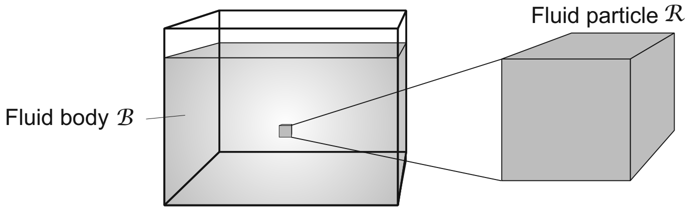


Figure 1.1.: A body $\mathcal{B}$ composed of particles $\mathcal{R}$.

temperature T, inner energy e and so on are functions of space and time, and are written as density $\rho(x_i, t)$, pressure $p(x_i, t)$, velocity $\boldsymbol{v}(x_i, t)$, temperature $T(x_i, t)$, inner energy $e(x_i, t)$, etc.. So, the total change of a scalar quantity like the density ρ is

$$\mathrm{d}\rho = \left(\frac{\partial \rho}{\partial t}\right) \mathrm{d}t + \left(\frac{\partial \rho}{\partial x_1}\right) \mathrm{d}x_1 + \left(\frac{\partial \rho}{\partial x_2}\right) \mathrm{d}x_2 + \left(\frac{\partial \rho}{\partial x_3}\right) \mathrm{d}x_3 \,. \tag{1.1}$$

Therefore, the total derivative (also called substantial derivative) computes by

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_1}\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right) + \frac{\partial \rho}{\partial x_2}\left(\frac{\mathrm{d}x_2}{\mathrm{d}t}\right) + \frac{\partial \rho}{\partial x_3}\left(\frac{\mathrm{d}x_3}{\mathrm{d}t}\right) \\ &= \frac{\partial \rho}{\partial t} + \sum_{1}^{3} \frac{\partial \rho}{\partial x_i}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i}\underbrace{\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)}_{v_i} \,. \end{aligned} \tag{1.2}$$

Note that in the last line of (1.2) we have used the summation rule of Einstein[1]. Furthermore, in literature the substantial derivative of a physical quantity is mainly denoted by the capital letter D and writes as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \,. \tag{1.3}$$

1.1. Spatial Reference Systems

A spatial reference system defines how the motion of a continuum is described i.e., from which perspective an observer views the matter. In a Lagrangian frame of reference, the

[1]In the following, we use both vector and index notation; for details see App. A and B.

observer monitors the trajectory in space of each material point and measures its physical quantities. This can be understood by considering a measuring probe which moves together with the material, like a boat on a river. The advantage is that free or moving boundaries can be captured easily as they require no special effort. Therefore, the approach is suitable in the case of structural mechanics. However, its limitation is obtained dealing with large deformation, as in the case of fluid dynamics. In this case, a better choice is the Eulerian frame of reference, in which the observer monitors a single point in space when measuring physical quantities – the measuring probe stays at a fixed position in space. However, contrary to the Lagrangian approach, difficulties arise with deformations on the domain boundary, e.g., free boundaries and moving interfaces.

Formally, a deformation of a material body $\mathcal{B}$ is defined as a map ψ, which projects each point $\boldsymbol{X}$ at time $t \in \mathbb{R}$ to its current location $\boldsymbol{x}$, in mathematical terms

$$\boldsymbol{x} = \psi(\boldsymbol{X}, t), \qquad \psi : \mathcal{B} \times \mathbb{R} \to \mathbb{R}^3 \ .$$

By coupling structural and fluid mechanics an additional map between the different reference systems is necessary. In [1] a first method to solve the problem for an incompressible, viscous fluid has been presented. The so called Arbitrary-Lagrangian-Eulerian (ALE) method combines the advantages of both approaches. The concept is that the observer is neither fixed nor does move together with the material, but can move *arbitrarily*. Between each of the two reference systems a bijective mapping of the spatial variables $\boldsymbol{x}$ (Eulerian system), $\boldsymbol{X}$ (Lagrangian system) and $\boldsymbol{\chi}$ (ALE system) exists, as illustrated in Fig. 1.2. The choice of

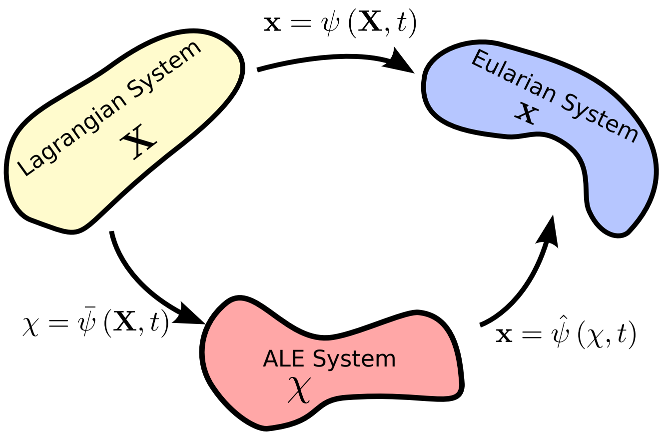


Figure 1.2.: Illustration of mapping between reference systems.

reference system effects the partial differential equations (PDEs) through its time derivative. Exemplified for a quantity f and its velocity $\boldsymbol{v}$, the total derivative results for the

- Lagrangian system to

$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{X}}$$

- Eulerian system to

$$\frac{Df}{Dt} = \underbrace{\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}}}_{\text{local change}} + \underbrace{(\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}})\, f}_{\text{convective change}}$$

- ALE system to
$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + (\boldsymbol{v}_c \cdot \nabla_{\boldsymbol{\chi}})\, f, \tag{1.4}$$
with the convective velocity $\boldsymbol{v}_c = \boldsymbol{v} - \boldsymbol{v}_g$, the difference between material velocity $\boldsymbol{v}$ and grid velocity $\boldsymbol{v}_g$.

1.2. Reynolds' Transport Theorem

To derive the integral form of the balance equations the rate of change of integrals of scalar and vector functions has to be described, which is known as the Reynolds' transport theorem. The volume integral can change for two reasons: (1) scalar or vector functions change (2) the volume changes. The following discussion is directed to scalar valued functions.

Let's consider a scalar quantity $f(\boldsymbol{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$, the change in time in a Lagrangian system of its volume integral
$$F(t) := \int\limits_{\Omega(t)} f(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \tag{1.5}$$
is given as
$$\frac{D}{Dt} F(t) = \frac{D}{Dt} \int\limits_{\Omega_L} f(\boldsymbol{X}, t)\, \mathrm{d}\boldsymbol{x} = \int\limits_{\Omega_L} \frac{\partial}{\partial t} f(\boldsymbol{X}, t)\, \mathrm{d}\boldsymbol{x}\,. \tag{1.6}$$
The simple transformation is due to the linearity of the integral and differential operators, and since the Lagrangian domain Ω_L conforms with the material movement, no additional terms are needed.
In an Eulerian context, time derivation must also take the time dependent domain $\Omega(t)$ into account by adding a surface flux term, which can be formulated as a volume term using the integral theorem of Gauß. This results in
$$\begin{aligned} \frac{D}{Dt} \int\limits_{\Omega(t)} f\, \mathrm{d}\boldsymbol{x} &= \int\limits_{\Omega(t)} \frac{\partial}{\partial t} f\, \mathrm{d}\boldsymbol{x} + \int\limits_{\Gamma(t)} f \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}s \\ &= \int\limits_{\Omega(t)} \left(\frac{\partial}{\partial t} f + \nabla \cdot (f \boldsymbol{v}) \right) \mathrm{d}\boldsymbol{x}\,. \end{aligned} \tag{1.7}$$

1.3. Conservation Equations

The basic equations for the flow field are the conservation of mass, momentum and energy. Together with the constitutive equations and equations of state, a full set of PDEs is derived.

1.3.1. Conservation of Mass

The mass m of a body is the volume integral of its density ρ,
$$m = \int\limits_{\Omega(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}\,. \tag{1.8}$$

Mass conservation states that the mass of a body is conserved over time, assuming there is no source or drain. Therefore, applying Reynolds' transport theorem (1.7), results in

$$\begin{aligned} \frac{Dm}{Dt} &= \int\limits_{\Omega} \frac{\partial \rho}{\partial t} \, \mathrm{d}\boldsymbol{x} + \int\limits_{\Gamma} \rho \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}s \\ &= \int\limits_{\Omega} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \right) \mathrm{d}\boldsymbol{x} = 0. \end{aligned} \tag{1.9}$$

The integral in (1.9) can be dismissed, as it holds for arbitrary Ω and in the special case of an incompressible fluid ($\rho = \mathit{const.} \quad \forall (\boldsymbol{x}, t) \in \Omega \times \mathbb{R}$), which may be assumed for low Mach numbers (see Sec. 1.4), the time and space derivative of the density vanishes. This lead to the following form of mass conservation equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \qquad \text{(compressible fluid)}, \\ \nabla \cdot \boldsymbol{v} &= 0 \qquad \text{(incompressible fluid)}. \end{aligned} \tag{1.10}$$

1.3.2. Conservation of Momentum

The equation of momentum is implied by Newtons second law and states that momentum $\boldsymbol{I}_{\mathrm{m}}$ is the product of mass m and velocity $\boldsymbol{v}$

$$\boldsymbol{I}_{\mathrm{m}} = m\boldsymbol{v}\,. \tag{1.11}$$

Derivation in time gives the rate of change of momentum, which is equal to the force $\boldsymbol{F}$ and reveals the relation to Newtons second law in an Eulerian reference system

$$\boldsymbol{F} = \frac{D\boldsymbol{I}_{\mathrm{m}}}{Dt} = \frac{D}{Dt}(m\boldsymbol{v}) = \frac{\partial}{\partial t}(m\boldsymbol{v}) + \nabla \cdot (m\boldsymbol{v} \otimes \boldsymbol{v}), \tag{1.12}$$

where $\boldsymbol{v} \otimes \boldsymbol{v}$ is a tensor defined by the dyadic product $\otimes$ (see App. B). The last equality in (1.12) is derived from Reynolds transport theorem (1.7) and mass conservation (1.10).

The forces $\boldsymbol{F}$ acting on fluids can be split up into forces acting on the surface of the body $\boldsymbol{F}_{\Gamma}$, forces due to momentum of the molecules $D\boldsymbol{I}_{\mathrm{m}}/Dt$ and external forces $\boldsymbol{F}_{\mathrm{ex}}$ (e.g. gravity, electromagnetic forces)

$$\boldsymbol{F} = \boldsymbol{F}_{\Gamma} + \frac{D}{Dt}\boldsymbol{I}_{\mathrm{m}} + \boldsymbol{F}_{\mathrm{ex}}. \tag{1.13}$$

Thereby, the surface force computes by

$$\sum_{i=1}^{3} \boldsymbol{F}_{\Gamma_j} = -\sum_{i=1}^{3} \frac{\partial p}{\partial x_j} \Omega \boldsymbol{n}_j = -\Omega \nabla p. \tag{1.14}$$

and the total change of momentum $\boldsymbol{I}_{\mathrm{m}}$ by

$$\frac{D}{Dt}\boldsymbol{I}_{\mathrm{m}} = \Omega \, \nabla \cdot [\boldsymbol{\tau}] \tag{1.15}$$

with the viscous stress tensor $[\boldsymbol{\tau}]$ (see Fig. 1.3).

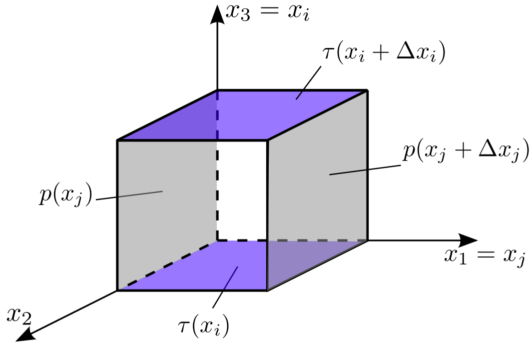


Figure 1.3.: Forces acting on a fluid element.

Now, we exploit the fact that $m = \rho\Omega$ and insert the pressure force (1.14), the viscous force (1.15) and any external forces per unit volume $\boldsymbol{f}$ acting on the fluid into (1.12). Thereby, we arrive at the momentum equation

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \nabla \cdot [\boldsymbol{\tau}] + \boldsymbol{f} \tag{1.16}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{v} \otimes \boldsymbol{v} + p\, [\boldsymbol{I}] - [\boldsymbol{\tau}]) = \boldsymbol{f} \tag{1.17}$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_j v_i + p\delta_{ij} - \tau_{ij}) = f_i\,, \tag{1.18}$$

with $[\boldsymbol{I}]$ the identity tensor. Furthermore, we introduce the momentum flux tensor $[\boldsymbol{\pi}]$ defined by

$$\pi_{ij} = \rho v_i v_j + p\delta_{ij} - \tau_{ij}\,, \tag{1.19}$$

and the fluid stress tensor $[\boldsymbol{\sigma}_{\mathrm{f}}]$ by

$$[\boldsymbol{\sigma}_{\mathrm{f}}] = -p\, [\boldsymbol{I}] + [\boldsymbol{\tau}]\,. \tag{1.20}$$

To arrive at an alternative formulation for the momentum equation, also called the non-conservative form, we exploit the following identities

$$\nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \boldsymbol{v}\, \nabla \cdot (\rho \boldsymbol{v}) \tag{1.21}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} = \rho \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \frac{\partial \rho}{\partial t} \tag{1.22}$$

and rewrite (1.16) by

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\, \frac{\partial \rho}{\partial t} + \boldsymbol{v} \nabla \cdot (\rho \boldsymbol{v}) + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot [\boldsymbol{\tau}] + \boldsymbol{f}\,. \tag{1.23}$$

Now, we use the mass conservation and arrive at

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot [\boldsymbol{\tau}] + \boldsymbol{f} \tag{1.24}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i\,.$$

1.3.3. Conservation of energy

The total balance of energy considers the inner, the kinetic and potential energies of a fluid. Since we do not consider gravity, the total change of energy over time for a fluid element with mass m is given by

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(m\left(\frac{1}{2}v^2+e\right)\right) = m\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v^2+e\right) + \left(\frac{1}{2}v^2+e\right)\frac{\mathrm{D}m}{\mathrm{D}t} \tag{1.25}$$

with e the inner energy and $v^2 = \boldsymbol{v}\cdot\boldsymbol{v}$. Due to mass conservation, the second term is zero and we obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(m\left(\frac{1}{2}v^2+e\right)\right) = \rho\Omega\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v^2+e\right). \tag{1.26}$$

This change of energy can be caused by [2]

- heat production per unit of volume: $q_\mathrm{h}\,\Omega$
- heat conduction energy due to heat flux $\boldsymbol{q}_\mathrm{T}$: $(-\partial q_{\mathrm{T}i}/\partial x_i)\,\Omega$
- energy due to surface pressure force: $(-\partial/\partial x_i(pv_i))\,\Omega$
- energy due to surface shear force: $(-\partial/\partial x_i(\tau_{ij}v_j))\,\Omega$
- mechanical energy due to the force density $\boldsymbol{f}_i$ given by: $(f_i v_i)\,\Omega$

Thereby, we arrive at the conservation of energy given by

$$\rho\,\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v^2+e\right) = -\frac{\partial q_{\mathrm{T}i}}{\partial x_i} - \frac{\partial p v_i}{\partial x_i} - \frac{\partial \tau_{ij}v_j}{\partial x_i} + f_i v_i + q_\mathrm{h} \tag{1.27}$$

or in vector notation by

$$\rho\,\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v^2+e\right) = -\nabla\cdot\boldsymbol{q}_\mathrm{T} - \nabla\cdot(p\boldsymbol{v}) - \nabla\cdot([\boldsymbol{\tau}]\cdot\boldsymbol{v}) + \boldsymbol{f}\cdot\boldsymbol{v} + q_\mathrm{h}\,. \tag{1.28}$$

By further exploring thermodynamic relations (see Sec. 1.3.4) and the mechanical energy (obtained by inner product of momentum conservation with $\boldsymbol{v}$), we may write (1.28) by the specific entropy s as follows [3]

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t} = \tau_{ij}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_{\mathrm{T}i}}{\partial x_i} + q_\mathrm{h}\,. \tag{1.29}$$

When heat transfer is neglected, the flow ist *adiabatic*. It is *isentropic*, when it is adiabatic and reversible, which means that the viscous dissipation can be neglected, which leads to (no heat production)

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t} = 0\,. \tag{1.30}$$

Finally, when the fluid is homogeneous and the entropy uniform ($\mathrm{D}s = 0$), we call the flow *homentropic*.

1.3.4. Constitutive equations

The conservation of mass, momentum and energy involve much more unknowns than equations. To close the system, additional information is provided by empirical information in form of constitutive equations. A good approximation is obtained by assuming the fluid to be in thermodynamic equilibrium. This implies for a homogeneous fluid that two *intrinsic* state variables fully determine the state of the fluid.

When we apply specific heat production q_h to a fluid element, then the specific inner energy e increases and at the same time the volume changes by $p\,\mathrm{d}\rho^{-1}$. This thermodynamic relation is expressed by

$$\mathrm{d}e = \mathrm{d}q_\mathrm{h} - p\,\mathrm{d}\rho^{-1}\,, \tag{1.31}$$

where the second term describes the work done on the fluid element by the pressure. If the change occurs sufficiently slowly, the fluid element is always in thermodynamic equilibrium, and we can express the heat input by the specific entropy s

$$\mathrm{d}q_\mathrm{h} = T\,\mathrm{d}s\,. \tag{1.32}$$

Therefore, we may rewrite (1.31) and arrive at the fundamental law of thermodynamics

$$\begin{aligned}\mathrm{d}e &= T\,\mathrm{d}s - p\,\mathrm{d}\rho^{-1}\\ &= T\,\mathrm{d}s + \frac{p}{\rho^2}\,\mathrm{d}\rho\,.\end{aligned} \tag{1.33}$$

Towards acoustics, it is convenient to choose the mass density ρ and the specific entropy s as intrinsic state variables. Hence, the specific inner energy e is completely defined by a relation denoted as the thermal equation of state

$$e = e(\rho,\, s)\,. \tag{1.34}$$

Therefore, variations of e are given by

$$de = \left(\frac{\partial e}{\partial \rho}\right)_s \mathrm{d}\rho + \left(\frac{\partial e}{\partial s}\right)_\rho \mathrm{d}s\,. \tag{1.35}$$

A comparison with the fundamental law of thermodynamics (1.33) provides the thermodynamic equations for the temperature T and pressure p

$$T = \left(\frac{\partial e}{\partial s}\right)_\rho\,; \quad p = \rho^2 \left(\frac{\partial e}{\partial \rho}\right)_s\,. \tag{1.36}$$

Since p is a function of ρ and s, we may write

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s \mathrm{d}\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \mathrm{d}s\,. \tag{1.37}$$

As sound is defined as isentropic ($ds = 0$) pressure-density perturbations, the isentropic speed of sound is defined by

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}\,. \tag{1.38}$$

Since in many applications the fluid considered is air at ambient pressure and temperature, we may use the ideal gas law

$$p = \rho R T \tag{1.39}$$

with the specific gas constant R, which computes for an ideal gas as

$$R = c_p - c_\Omega \,. \tag{1.40}$$

In (1.40) c_p, c_Ω denote the specific heat at constant pressure and constant volume, respectively. Furthermore, the inner energy e depends for an ideal gas just on the temperature T via

$$\mathrm{d}e = c_\Omega \, \mathrm{d}T \,. \tag{1.41}$$

Substituting this relations in (1.33), assuming an isentropic state ($\mathrm{d}s = 0$) and using (1.39) results in

$$c_\Omega \, \mathrm{d}T = \frac{p}{\rho^2} \, \mathrm{d}\rho \;\rightarrow\; \frac{\mathrm{d}T}{T} = \frac{R}{c_\Omega} \frac{\mathrm{d}\rho}{\rho} \,. \tag{1.42}$$

dividing by p and directly using (1.39) results in

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T} \,. \tag{1.43}$$

This relation and applying (1.42), (1.40) leads to

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}\rho}{\rho} + \frac{R}{c_\Omega} \frac{\mathrm{d}\rho}{\rho} = \frac{c_p}{c_\Omega} \frac{\mathrm{d}\rho}{\rho} = \kappa \, \frac{\mathrm{d}\rho}{\rho} \tag{1.44}$$

with κ the specific heat ratio (also known as adiabatic exponent). A comparison of (1.44) with (1.38) yields

$$c = \sqrt{\kappa p / \rho} = \sqrt{\kappa R T} \,. \tag{1.45}$$

We see that the speed of sound c of an ideal gas depends only on the temperature. For air κ has a value of 1.402 so that we obtain a speed of sound c at $T = 15°\mathrm{C}$ of 341 m/s. For most practical applications, we can set the speed of sound to 340 m/s within a temperature range of 5°C to 25°C. Combining (1.44) and (1.45), we obtain the general pressure-density relation for an isentropic state

$$\frac{\mathrm{d}p}{\mathrm{d}t} = c^2 \frac{\mathrm{d}\rho}{\mathrm{d}t} \,. \tag{1.46}$$

Furthermore, since we use an Eulerian frame of reference, we may rewrite (1.46) by

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \,. \tag{1.47}$$

As we consider local thermodynamic equilibrium, it is reasonable to assume that transport processes are determined by linear functions of the gradient of the flow state variables. This corresponds to *Newtonian fluid* behavior expressed by

$$\tau_{ij} = 2\mu \, \epsilon_{ij} + \lambda \epsilon_{ii} \delta_{ij} \tag{1.48}$$

with the rate of the strain tensor $\boldsymbol{\epsilon}$

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j} \right) \,. \tag{1.49}$$

Note that the term $\epsilon_{ii} = \nabla \cdot \boldsymbol{v}$ takes into account the effect of dilatation. In thermodynamic equilibrium, the bulk viscosity λ is equal to $-(2/3)\mu$ (with μ being the dynamic viscosity) according to the hypothesis of Stokes, and we may write

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial v_k}{\partial x_k} \delta_{ij} \,. \tag{1.50}$$

With this relation, we can also rewrite the divergence of the $[\boldsymbol{\tau}]$ as

$$\nabla \cdot [\tau] = \mu \left(\nabla \cdot \nabla \boldsymbol{v} + \frac{1}{3} \nabla \nabla \cdot \boldsymbol{v} \right) \,. \tag{1.51}$$

1.4. Characterization of Flows by Dimensionless Numbers

Two flows around geometric similar models are physically similar if all characteristic numbers coincide [4]. Especially for measurement setups, these similarity considerations are important as it allows measuring of down sized geometries. Furthermore, the characteristic numbers are used to classify a flow situation. The Reynolds number is defined by

$$\mathrm{Re} = \frac{vl}{\nu} \tag{1.52}$$

with the characteristic flow velocity v, flow length l and kinematic viscosity ν. It provides the ratio between stationary inertia forces and viscous forces. Thereby, it allows to subdivide flows into laminar and turbulent ones. The Mach number allows for an approximative subdivision of a flow in compressible ($\mathrm{Ma} > 0.3$) and incompressible ($\mathrm{Ma} \leq 0.3$), and is defined by

$$\mathrm{Ma} = \frac{v}{c} \tag{1.53}$$

with c the speed of sound. In unsteady problems, periodic oscillating flow structures may occur, e.g. the Kármán vortex street in the wake of a cylinder. The dimensionless frequency of such an oscillation is denoted as the Strouhal number, and is defined by

$$\mathrm{St} = f \, \frac{l}{v} \tag{1.54}$$

with f the shedding frequency.

1.5. Towards Acoustics

According to the Helmholtz decomposition, the velocity vector $\boldsymbol{v}$ (as any vector field) can be split into an irrotational part and a solenoidal part

$$\boldsymbol{v} = \nabla \phi + \nabla \times \boldsymbol{\Psi} \,, \tag{1.55}$$

where ϕ is a scalar potential and $\boldsymbol{\Psi}$ a vector potential. Thereby, we call a flow being purely described by a scalar potential via

$$\boldsymbol{v} = \nabla \phi$$

a *potential flow*. Using (1.55), mass conservation (see (1.10)) may be written as

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) &= \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{v} \\
&= \frac{D\rho}{Dt} + \rho \nabla \cdot \nabla \phi + \rho \underbrace{\nabla \cdot \nabla \times \boldsymbol{\Psi}}_{=0} \\
\frac{1}{\rho}\frac{D\rho}{Dt} &= -\nabla \cdot \nabla \phi \,. \qquad (1.56)
\end{aligned}$$

This result obviously leads us to the interpretation that the flow related to the acoustic field is an irrotational flow and that the acoustic field is the unsteady component of the gradient of the velocity potential ϕ. On the other hand, taking the curl of (1.55) results in the vorticity of the flow

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \nabla \times \nabla \times \boldsymbol{\Psi} + \nabla \times \nabla \phi = \nabla \times \nabla \times \boldsymbol{\Psi} \,. \qquad (1.57)$$

We see that this quantity is fully defined by the vector potential and characterizes the solenoidal part of the flow field.

Let us consider a pulsating sphere as displayed in Fig. 1.4. Since there are no sources in

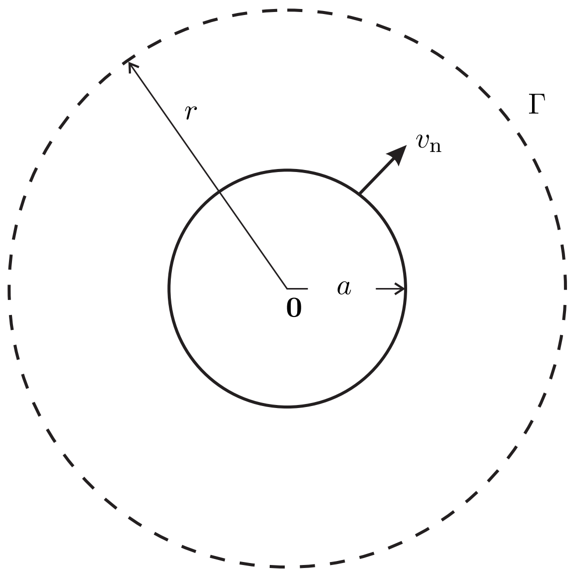


Figure 1.4.: Pulsating sphere.

the fluid and we assume the fluid to be incompressible, we can model it by the Laplacian of the scalar velocity potential

$$\nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \,. \qquad (1.58)$$

Since the setup is radially symmetric, we obtain (using spherical coordinates)

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\phi = 0\,; \quad r > a$$

and hence

$$\phi = \frac{A}{r} + B \,.$$

We assume that ϕ vanishes at ∞, so B can be set to zero. Furthermore, with the boundary condition $\partial\phi/\partial r = v_\mathrm{n}$ at $r = a$ we get

$$\phi(t) = -\frac{a^2}{r}\, v_\mathrm{n}(t) \text{ for } r > a\,. \tag{1.59}$$

Assuming a non-viscous fluid ($[\boldsymbol{\tau}] = \mathbf{0}$), no external forces ($\boldsymbol{f} = \mathbf{0}$) and neglecting the convective term, we may write the momentum conservation (see (1.24)) for an incompressible flow by

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} + \nabla p = 0\,. \tag{1.60}$$

Using the scalar potential ϕ, we arrive at the following linearized relation

$$p = -\rho_0 \frac{\partial \phi}{\partial t}\,. \tag{1.61}$$

With (1.59) we can compute the resulting pressure p to

$$p(t) = -\rho_0 \frac{\partial \phi}{\partial t} = \rho_0\, \frac{a^2}{r}\, \frac{\partial v_\mathrm{n}}{\partial t}\,. \tag{1.62}$$

The volume flux $q_\Omega(t)$ at any time computes as

$$q_\Omega(t) = \oint_\Gamma \nabla\phi \cdot \mathrm{d}\boldsymbol{s} = \oint_\Gamma \underbrace{\nabla\phi \cdot \boldsymbol{e}_\mathrm{r}}_{\partial\phi/\partial r = v_\mathrm{n}}\ \mathrm{d}s = 4\pi a^2\, v_\mathrm{n}(t)\,,$$

and so we can rewrite (1.59) by

$$\phi(t) = -\frac{q_\Omega(t)}{4\pi r} \text{ for } r > a\,. \tag{1.63}$$

This solution also holds for $r \to 0$ (see [5]).

1.6. Questions: Chapter 1

1. Explain the difference between the Eulerian and the Lagrangian description of physical fields; provide at least three examples.

2. Use Reynolds transport theorem

$$\frac{D}{Dt} \int_\Omega \gamma(\boldsymbol{x}, t)\ \mathrm{d}\boldsymbol{x} = \int_\Omega \left(\left.\frac{\partial \gamma(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot (\gamma \boldsymbol{v}) \right)\ \mathrm{d}\boldsymbol{x}$$

   to arrive the conservation of mass for fluids.

3. The conservation of momentum is given by

$$\rho\, \frac{\partial \boldsymbol{v}}{\partial t} + \rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \boldsymbol{f}$$

   Rewrite this equation with Einstein's index notation.

4. How are *insentropic* and *homentropic* state defined?
5. When a fluid is at a thermodynamic equilibrium, we can define the state of the fluid by two intrinsic variables. Towards acoustics, what are the two intrinsic quantities?
6. What tells us the Helmholtz decomposition? Apply it to the flow velocity and discuss the individual terms.
7. Show that the compressibility of a flow can be described by a scalar potential.
8. What is the vorticity and how is it computed?

2. Acoustics

2.1. Wave equation

We assume an isentropic case, where the total variation of the entropy is zero and the pressure is only a function of the density. For linear acoustics, this results in the well known relation between the acoustic pressure p_a and density ρ_a

$$p_\mathrm{a} = c_0^2 \rho_\mathrm{a} \tag{2.1}$$

with a constant speed of sound c_0. Furthermore, the acoustic field can be seen as a perturbation of the mean flow field

$$p = p_0 + p_\mathrm{a}\,; \quad \rho = \rho_0 + \rho_\mathrm{a}\,; \quad \boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_\mathrm{a} \tag{2.2}$$

with the following relations

$$p_\mathrm{a} \ll p_0\,; \quad \rho_\mathrm{a} \ll \rho_0\,. \tag{2.3}$$

Furthermore, we assume the viscosity to be zero, so that the viscous stress tensor $[\boldsymbol{\tau}]$ can be neglected, and the force density $\boldsymbol{f}$ is zero. We call ρ_a the acoustic density and $\boldsymbol{v}_\mathrm{a}$ the acoustic particle velocity.

For a quiescent fluid, the mean velocity $\boldsymbol{v}_0$ is zero, and furthermore we assume a spatial and temporal constant mean density ρ_0 and pressure p_0. Using the perturbation ansatz (2.2) and substituting it into (1.10) and (1.24), results in

$$\frac{\partial(\rho_0 + \rho_\mathrm{a})}{\partial t} + \nabla \cdot \Big((\rho_0 + \rho_a)\boldsymbol{v}_\mathrm{a} \Big) = 0 \tag{2.4}$$

$$(\rho_0 + \rho_\mathrm{a}) \frac{\partial \boldsymbol{v}_\mathrm{a}}{\partial t} + \Big((\rho_0 + \rho_\mathrm{a})\boldsymbol{v}_\mathrm{a} \Big) \cdot \nabla \boldsymbol{v}_\mathrm{a} = -\nabla \big(p_0 + p_\mathrm{a} \big)\,. \tag{2.5}$$

In a next step, since we derive linear acoustic conservation equations, we are allowed to cancel second order terms (e.g., such as $\rho_\mathrm{a}\boldsymbol{v}_\mathrm{a}$), and arrive at conservation of mass and momentum

$$\frac{\partial \rho_\mathrm{a}}{\partial t} + \rho_0 \nabla \boldsymbol{v}_\mathrm{a} = 0 \tag{2.6}$$

$$\rho_0 \frac{\partial \boldsymbol{v}_\mathrm{a}}{\partial t} + \nabla p_\mathrm{a} = \boldsymbol{0}\,. \tag{2.7}$$

Applying the curl-operation to (2.7) results in

$$\nabla \times \frac{\partial \boldsymbol{v}_\mathrm{a}}{\partial t} = \boldsymbol{0}\,, \tag{2.8}$$

which allows us to introduce the scalar acoustic potential ψ_a via

$$\boldsymbol{v}_\mathrm{a} = -\nabla \psi_\mathrm{a}\,. \tag{2.9}$$

Substituting (2.9) into (2.7) results in the well known relation between acoustic pressure and scalar potential

$$p_\mathrm{a} = \rho_0 \frac{\partial \psi_\mathrm{a}}{\partial t}\,. \tag{2.10}$$

Now, we substitute this relation into (2.6), use (2.1) and arrive at the well known acoustic wave equation

$$\frac{1}{c_0^2}\frac{\partial^2 \psi_\mathrm{a}}{\partial t^2} - \Delta \psi_\mathrm{a} = 0\,. \tag{2.11}$$

On the other hand, we also obtain the wave equation for the acoustic pressure p_a exploring (2.4), (2.5) and (2.1)

$$\frac{1}{c_0^2}\frac{\partial^2 p_\mathrm{a}}{\partial t^2} - \Delta p_\mathrm{a} = 0\,. \tag{2.12}$$

2.2. Compactness

In regions, e.g. at boundaries, where the acoustic potential ψ_a varies significantly over a distance l, which is short compared to the wavelength λ, the acoustic field can be approximated by the incompressible potential flow. We call such a region *compact*, and a source size much smaller than λ is a *compact source*. For a precise definition, we define a typical time scale τ (or angular frequency f_c) and a length scale l_c. Then, the dimensionless form of the wave equation reads

$$\frac{\partial^2 \psi_\mathrm{a}}{\partial \tilde{x}_i^2} = \mathrm{He}^2 \frac{\partial^2 \psi_\mathrm{a}}{\partial \tilde{t}^2} \tag{2.13}$$

with $\tilde{t} = t/\tau = \omega t$ and $\tilde{x}_i = x_i/l_\mathrm{c}$. In (2.13) He denotes the Helmholtz number and computes by

$$\mathrm{He} = \frac{l_\mathrm{c}}{c_0 \tau} = \frac{f_\mathrm{c} l_\mathrm{c}}{c_0} = \frac{2\pi l_\mathrm{c}}{\lambda_\mathrm{c}} \ll 1\,.$$

Note that the time derivative term in (2.13) is multiplied by the square of a Helmholtz-number. Therefore, if He is small, we may neglect this term and the wave equation reduces to

$$\nabla \cdot \nabla \psi_\mathrm{a} = 0\,. \tag{2.14}$$

Hence, we can describe the acoustic field by the incompressible potential flow, which allows us to use incompressible potential flow theory to derive the local behavior of acoustic fields in compact regions.

2.3. Simple solutions

In order to get some physical insight in the propagation of acoustic sound, we will consider two special cases: plane and spherical waves. Let's start with the simpler case, the propagation of a plane wave as displayed in Fig. 2.1. Thus, we can express the acoustic pressure by $p_\mathrm{a} = p_\mathrm{a}(x,t)$ and the particle velocity by $\boldsymbol{v}_\mathrm{a} = v_\mathrm{a}(x,t)\boldsymbol{e}_x$. Using these relations together with the linear pressure-density law (assuming constant mean density, see (2.1)), we arrive at the following 1D linear wave equation

$$\frac{\partial^2 p_\mathrm{a}}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p_\mathrm{a}}{\partial t^2} = 0\,, \tag{2.15}$$

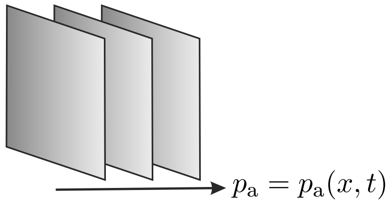


Figure 2.1.: Propagation of a plane wave.

which can be rewritten in factorized version as

$$\left(\frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{c_0}\frac{\partial}{\partial t}\right) p_\mathrm{a} = 0\,. \tag{2.16}$$

This version of the linearized, 1D wave equation motivates us to introduce the following two functions (solution according to d'Alembert)

$$\xi \;=\; t - x/c_0\,; \quad \eta = t + x/c_0$$

with properties

$$\frac{\partial}{\partial t} \;=\; \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\,; \qquad \frac{\partial}{\partial x} = \frac{1}{c_0}\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)\,.$$

Therewith, we obtain for the factorized operator

$$\frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t} \;=\; -\frac{2}{c_0}\frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial x} + \frac{1}{c_0}\frac{\partial}{\partial t} = \frac{2}{c_0}\frac{\partial}{\partial \eta}$$

and the linear, 1D wave equation transfers to

$$-\frac{4}{c_0^2}\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} p_\mathrm{a} = 0\,.$$

The general solution computes as a superposition of arbitrary functions of ξ and η

$$p_\mathrm{a} = f(\xi) + f(\eta) = f(t - x/c_0) + g(t + x/c_0)\,. \tag{2.17}$$

This solution describes waves moving with the speed of sound c_0 in $+x$ and $-x$ direction, respectively. In a next step, we use the linearized conservation of momentum according to (2.5), and rewrite it for the 1D case (assuming zero source term)

$$\rho_0 \frac{\partial v_\mathrm{a}}{\partial t} + \frac{\partial p_\mathrm{a}}{\partial x} = 0\,. \tag{2.18}$$

Now, we use just consider a forward propagating wave, i.e. $g(t) = 0$, substitute (2.17) into (2.18) and obtain

$$\begin{aligned} v_\mathrm{a} \;&=\; -\frac{1}{\rho_0}\int \frac{\partial p_\mathrm{a}}{\partial x}\,\mathrm{d}t = \frac{1}{\rho_0 c_0}\int \frac{\partial f(t - x/c_0)}{\partial t}\,\mathrm{d}t \\ &=\; \frac{1}{\rho_0 c_0} f(t - x/c_0) = \frac{p_\mathrm{a}}{\rho_0 c_0}\,. \end{aligned} \tag{2.19}$$

Therewith, the value of the acoustic pressure over acoustic particle velocity for a plane wave is constant. To allow for a general orientation of the coordinate system, a free field plane wave may be expressed by

$$p_{\mathrm{a}} = f(\boldsymbol{n} \cdot \boldsymbol{x} - c_0 t)\,; \quad \boldsymbol{v}_{\mathrm{a}} = \frac{\boldsymbol{n}}{\rho_0 c_0}\, f(\boldsymbol{n} \cdot \boldsymbol{x} - c_0 t)\,, \tag{2.20}$$

where the direction of propagation is given by the unit vector $\boldsymbol{n}$. A time harmonic plane wave of angular frequency $\omega = 2\pi f$ is usually written as

$$p_{\mathrm{a}}\,,\ \boldsymbol{v}_{\mathrm{a}} \sim e^{j(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \tag{2.21}$$

with the wave number (also called wave vector) $\boldsymbol{k}$, which computes by

$$\boldsymbol{k} = k\boldsymbol{n} = \frac{\omega}{c_0}\,\boldsymbol{n}\,. \tag{2.22}$$

The second case of investigation will be a spherical wave, where we assume a point source located at the origin. In the first step, we rewrite the linearized wave equation in spherical coordinates and consider that the pressure p_{a} will just depend on the radius r. Therewith, the Laplace-operator reads as

$$\nabla \cdot \nabla p_{\mathrm{a}}(r,t) = \frac{\partial^2 p_{\mathrm{a}}}{\partial r^2} + \frac{2}{r}\frac{\partial p_{\mathrm{a}}}{\partial r} = \frac{1}{r}\frac{\partial^2 r p_{\mathrm{a}}}{\partial r^2}$$

and we obtain

$$\frac{1}{r}\frac{\partial^2 r p_{\mathrm{a}}}{\partial r^2} - \frac{1}{c_0^2}\underbrace{\frac{\partial^2 p_{\mathrm{a}}}{\partial t^2}}_{\frac{1}{r}\frac{\partial^2 r p_{\mathrm{a}}}{\partial t^2}} = 0\,. \tag{2.23}$$

A multiplication of (2.23) with r results in the same wave equation as obtained for the plane case (see (2.15)), just instead of p_{a} we have rp_{a}. Therefore, the solution of (2.23) reads as

$$p_{\mathrm{a}}(r,t) = \frac{1}{r}\,\left(f(t - r/c_0) + g(t + r/c_0)\right)\,, \tag{2.24}$$

which means that the pressure amplitude will decrease according to the distance r from the source. The assumed symmetry requires that all quantities will just exhibit a radial component. Therewith, we can express the time averaged acoustic intensity $\boldsymbol{I}_{\mathrm{a}}^{\mathrm{av}}$ in normal direction $\boldsymbol{n}$ by a scalar value just depending on r

$$\boldsymbol{I}_{\mathrm{a}}^{\mathrm{av}} \cdot \boldsymbol{n} = I_r^{\mathrm{av}}$$

and as a function of the time averaged acoustic power $P_{\mathrm{a}}^{\mathrm{av}}$ of our source

$$I_r^{\mathrm{av}} = \frac{P_{\mathrm{a}}^{\mathrm{av}}}{4\pi r^2}\,. \tag{2.25}$$

According to (2.25), the acoustic intensity decreases with the squared distance from the source. This relation is known as the *spherical spreading law.*

In order to obtain the acoustic velocity $\boldsymbol{v}_\mathrm{a} = v_\mathrm{a}(r,t)\boldsymbol{e}_r$ as a function of the acoustic pressure p_a, we substitute the general solution for p_a (see (2.24), in which we set without loss of generality $g = 0$) into the linear momentum equation (see (2.7))

$$\begin{aligned}\frac{\partial v_\mathrm{a}}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p_\mathrm{a}}{\partial r} = -\frac{1}{\rho_0}\frac{\partial}{\partial r}\left(\frac{f(t-r/c_0)}{r}\right) \\ v_\mathrm{a} &= -\frac{1}{\rho_0}\frac{\partial}{\partial r}\left(\frac{F(t-r/c_0)}{r}\right) \qquad (2.26)\end{aligned}$$

with $f(t) = \partial F(t)/\partial t$. Using the relation

$$\frac{\partial F(t-r/c_0)}{\partial r} = -\frac{1}{c_0}\frac{\partial F(t-r/c_0)}{\partial t}$$

and performing the differentiation with respect to r results in

$$\begin{aligned} v_\mathrm{a}(r,t) &= -\frac{1}{\rho_0}\frac{1}{r}\frac{\partial F(t-r/c_0)}{\partial r} + \frac{F(t-r/c_0)}{\rho_0 r^2} \qquad (2.27)\\ &= \frac{1}{\rho_0 c_0}\underbrace{\frac{1}{r}\frac{\partial F(t-r/c_0)}{\partial t}}_{f/r=p_\mathrm{a}} + \frac{F(t-r/c_0)}{\rho_0 r^2} \qquad (2.28)\\ &= \frac{p_\mathrm{a}}{\rho_0 c_0} + \frac{F(t-r/c_0)}{\rho_0 r^2}\,. \qquad (2.29)\end{aligned}$$

Therewith, spherical waves show in the limit $r \to \infty$ the same acoustic behavior as plane waves.

Now with this acoustic velocity-pressure relation, we may rewrite the acoustic intensity for spherical waves as

$$I_r = \frac{{p_\mathrm{a}}^2}{\rho_0 c_0} + \frac{p_\mathrm{a}}{\rho_0 r^2}F(t-r/c_0)$$

With the relation (just outgoing waves)

$$p_\mathrm{a} = \frac{f}{r} = \frac{1}{r}\frac{\partial F}{\partial t}$$

we obtain

$$I_r = \frac{{p_\mathrm{a}}^2}{\rho_0 c_0} + \frac{1}{2\rho_0 r^3}\frac{\partial F^2(t-r/c_0)}{\partial t}\,,$$

which results for the time averaged quantity (assuming $F(t-r/c)$ is a periodic function) in the same expression as for the plane wave

$$I_r^\mathrm{av} = \frac{({p_\mathrm{a}}^2)_\mathrm{av}}{\rho_0 c_0}\,.$$

2.4. Acoustic quantities and order of magnitudes

Let us consider a loudspeaker generating sound at a fixed frequency f and a number of microphones recording the sound as displayed in Fig. 2.2. In a first step, we measure the sound with one microphone fixed at $\boldsymbol{x}_0$, and we will obtain a periodic signal in time with

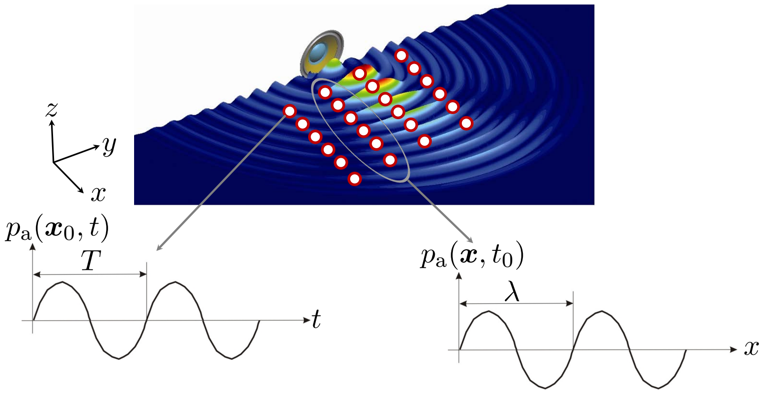


Figure 2.2.: Sound generated by a loudspeaker and measured by microphones.

the same frequency f and period time $T = 1/f$. In a second step, we use all microphones and record the pressure at a fixed time t_0. Drawing the obtained values along the individual positions of the microphone, e.g. along the coordinate x, we again obtain a periodic signal, which is now periodic in space. This periodicity is characterized by the wavelength λ and is uniquely defined by the frequency f and the speed of sound c_0 via the relation

$$\lambda = \frac{c_0}{f} \,. \tag{2.30}$$

Assuming a frequency of 1 kHz, the wavelength in air takes on the value of 0.343 m ($c_0 =$ 343 m/s).

Strictly speaking, each acoustic wave has to be considered as transient, having a beginning and an end. However, for some long duration sound, we speak of continuous wave (cw) propagation and we define for the acoustic pressure p_a a mean square pressure $(p_\mathrm{a})^2_\mathrm{av}$ as well as a root mean squared (rms) pressure $p_\mathrm{a,rms}$

$$p_\mathrm{a,rms} = \sqrt{\frac{1}{T} \int\limits_{t_0}^{t_0+T} (p - p_0)^2 \; \mathrm{d}t} = \sqrt{\frac{1}{T} \int\limits_{t_0}^{t_0+T} p_\mathrm{a}^2 \, \mathrm{d}t} \,. \tag{2.31}$$

In (2.31) T denotes the period time of the signal or if we cannot strictly speak of a periodic signal, an interminable long time interval. Now, it has to be mentioned that the threshold of hearing of an average human is at about 20 μPa and the threshold of pain at about 20 Pa, which differs 10^6 orders of magnitude. Thus, logarithmic scales are mainly used for acoustic quantities. The most common one is the *decibel* (dB), which expresses the quantity as a ratio relative to a reference value. Thereby, the sound pressure level $L_{\mathrm{p_a}}$ (SPL) is defined by

$$L_{\mathrm{p_a}} = 20 \, \log_{10} \frac{p_\mathrm{a,rms}}{p_\mathrm{a,ref}} \qquad p_\mathrm{a,ref} = 20 \, \mu\mathrm{P}a \,. \tag{2.32}$$

The reference pressure $p_{\mathrm{a},ref}$ corresponds to the sound at 1 kHz that an average person can just hear.

In addition, the acoustic intensity $\boldsymbol{I}_\mathrm{a}$ is defined by the product of the acoustic pressure and particle velocity

$$\boldsymbol{I}_\mathrm{a} = p_\mathrm{a}\boldsymbol{v}_\mathrm{a}\,. \tag{2.33}$$

The intensity level $L_{\mathrm{I_a}}$ is then defined by

$$L_{\mathrm{I_a}} = 10\,\log_{10}\frac{I_\mathrm{a}^\mathrm{av}}{I_\mathrm{a,ref}} \qquad I_\mathrm{a,ref} = 10^{-12}\,\mathrm{W/m^2}\,, \tag{2.34}$$

with $I_\mathrm{a,ref}$ the reference sound intensity corresponding to $p_\mathrm{a,ref}$. Again, we use an averaged value for defining the intensity level, which computes by

$$I_\mathrm{a}^\mathrm{av} = |\boldsymbol{I}_\mathrm{a}^\mathrm{av}| = \left|\frac{1}{T}\int\limits_{t_0}^{t_0+T}\boldsymbol{v}_\mathrm{a}p_\mathrm{a}\;\mathrm{d}t\,.\right| \tag{2.35}$$

Finally, we compute the acoustic power by integrating the acoustic intensity (unit $\mathrm{W/m^2}$) over a closed surface

$$P_\mathrm{a} = \oint\limits_\Gamma \boldsymbol{I}_\mathrm{a}\cdot\,\mathrm{d}\boldsymbol{s} = \oint\limits_\Gamma \boldsymbol{I}_\mathrm{a}\cdot\boldsymbol{n}\;\mathrm{d}s\,. \tag{2.36}$$

Then, the sound-power level $L_{\mathrm{P_a}}$ computes as

$$L_{\mathrm{P_a}} = 10\,\log_{10}\frac{P_\mathrm{a}^\mathrm{av}}{P_\mathrm{a,ref}} \qquad P_\mathrm{a,ref} = 10^{-12}\,\mathrm{W}\,, \tag{2.37}$$

with $P_\mathrm{a,ref}$ the reference sound power corresponding to $p_\mathrm{a,ref}$. In Tables 2.1 and 2.2 some typical sound pressure and sound power levels are listed.

Table 2.1.: Typical sound pressure levels SPL.

Threshold of hearing	Voice at 5 m	Car at 20 m	Pneumatic hammer at 2 m	Jet at 3 m
0 dB	60 dB	80 dB	100 dB	140 dB

Table 2.2.: Typical sound power levels and in parentheses the absolute acoustic power P_a.

Voice	Fan	Loudspeaker	Jet airliner
30 dB (25 μW)	110 dB (0.05 W)	128 dB (60 W)	170 dB (50 kW)

A useful quantity in the acoustics is impedance, which is a measure of the amount by which the motion induced by a pressure applied to a surface is impeded. However, a quantity that varies with time and depends on initial values is not of interest. Thus the specific acoustic impedance is defined via the Fourier transform by

$$\hat{Z}_{\rm a}(\boldsymbol{x},\omega) = \frac{\hat{p}_{\rm a}(\boldsymbol{x},\omega)}{\hat{\boldsymbol{v}}_{\rm a}(\boldsymbol{x},\omega)\cdot\boldsymbol{n}(\boldsymbol{x})} \tag{2.38}$$

at a point $\boldsymbol{x}$ on the surface Γ with unit normal vector $\boldsymbol{n}$. It is in general a complex number and its real part is called *resistance*, its imaginary part *reactance* and its inverse the *specific acoustic admittance* denoted by $\hat{Y}_{\rm a}(\boldsymbol{x},\omega)$. For a plane wave (see Sec. 2.3) the acoustic impedance $\hat{Z}_{\rm a}$ is constant

$$\hat{Z}_{\rm a}(\boldsymbol{x},\omega) = \rho_0 c_0\,. \tag{2.39}$$

For a quiescent fluid the acoustic power across a surface Γ computes for time harmonic fields by

$$\begin{aligned} P_{\rm a}^{\rm av} &= \int\limits_{\Gamma}\left(\frac{1}{T}\int\limits_0^T \mathrm{Re}\left(\hat{p}_{\rm a}e^{j\omega t}\right)\,\mathrm{Re}\left(\hat{\boldsymbol{v}}_{\rm a}\cdot\boldsymbol{n}e^{j\omega t}\right)\,\mathrm{d}t\right)\,\mathrm{d}s \\ &= \frac{1}{4}\int\limits_{\Gamma}\left(\hat{p}_{\rm a}\hat{\boldsymbol{v}}_{\rm a}^* + \hat{p}_{\rm a}^*\hat{\boldsymbol{v}}_{\rm a}\right)\cdot\boldsymbol{n}\,\mathrm{d}s \\ &= \frac{1}{2}\int\limits_{\Gamma}\mathrm{R}e\left(\hat{p}_{\rm a}^*\hat{\boldsymbol{v}}_{\rm a}\right)\cdot\boldsymbol{n}\,\mathrm{d}s \end{aligned} \tag{2.40}$$

with * denoting the conjugate complex. Now, we use the impedance $\hat{Z}_{\rm a}$ of the surface and arrive at

$$P_{\rm a}^{\rm av} = \frac{1}{2}\int\limits_{\Gamma}\mathrm{Re}\left(\hat{Z}_{\rm a}\right)\,|\hat{\boldsymbol{v}}_{\rm a}\cdot\boldsymbol{n}|^2\,\mathrm{d}s\,. \tag{2.41}$$

Hence, the real part of the impedance (equal to the resistance) is related to the energy flow. If $\mathrm{Re}\left(\hat{Z}_{\rm a}\right) > 0$ the surface is *passive* and absorbs energy, and if $\mathrm{Re}\left(\hat{Z}_{\rm a}\right) < 0$ the surface is *active* and produces energy.

In a next step, we analyze what happens, when an acoustic wave propagates from one fluid medium to another one. For simplicity, we restrict to a plane wave, which is described by (see (2.17))

$$p_{\rm a}(t) = f/t - x/c_0) + g(t + x/c_0) \tag{2.42}$$

In the frequency domain, we may write

$$\hat{p}_{\rm a} = \hat{f}e^{-j\omega x/c_0} + \hat{g}e^{j\omega x/c_0} = p^+e^{j\omega t - jkx} + p^-e^{j\omega t + jkx}\,. \tag{2.43}$$

Thereby, p^+ is the amplitude of the wave incident at $x = 0$ from $x < 0$ and p^- the amplitude of the reflected wave at $x = 0$ by an impedance $\hat{Z}_{\rm a}$. Using the linear conservation of momentum, we obtain the particle velocity

$$\hat{v}_{\rm a}(x) = \frac{1}{\rho_0 c_0}\left(p^+e^{-jkx} - p^-e^{jkx}\right)\,. \tag{2.44}$$

Defining the reflection coefficient R by

$$R = \frac{p^-}{p^+}\,, \tag{2.45}$$

we arrive with $\hat{Z}_\mathrm{a} = \hat{p}(0)/\hat{v}(0)$ at

$$R = \frac{\hat{Z}_\mathrm{a} - \rho_0 c_0}{\hat{Z}_\mathrm{a} + \rho_0 c_0}\,. \tag{2.46}$$

In two dimensions, we consider a plane wave with direction $(\cos\theta, \sin\theta)$, where θ is the angle with the y-axis and the wave approaches from $y < 0$ and hits an impedance $\hat{Z}$ at $y = 0$. The overall pressure may be expressed by

$$\hat{p}_\mathrm{a}(x,y) = e^{-jkx\sin\theta}\left(p^+ e^{-ky\cos\theta} + p^- e^{jky\cos\theta}\right)\,. \tag{2.47}$$

Furthermore, the y-component of the particle velocity computes to

$$\hat{v}_\mathrm{a}(x,y) = \frac{\cos\theta}{\rho_0 c_0} e^{-jkx\sin\theta}\left(p^+ e^{-ky\cos\theta} - p^- e^{jky\cos\theta}\right)\,. \tag{2.48}$$

Thereby, the impedance is

$$\hat{Z}_\mathrm{a} = \frac{\hat{p}(x,0)}{\hat{v}(x,0)} = \frac{\rho_0 c_0}{\cos\theta}\frac{p^+ + p^-}{p^+ - p^-} = \frac{\rho_0 c_0}{\cos\theta}\frac{1+R}{1-R} \tag{2.49}$$

so that the reflection coefficient computes as

$$R = \frac{\hat{Z}_\mathrm{a}\cos\theta - \rho_0 c_0}{\hat{Z}_\mathrm{a}\cos\theta + \rho_0 c_0}\,. \tag{2.50}$$

2.5. Impulsive sound sources

The sound being generated by a unit, impulsive point source $\delta(\boldsymbol{x})\delta(t)$ is the solution of

$$\frac{1}{c_0^2}\frac{\partial^2 \psi_\mathrm{a}}{\partial t^2} - \nabla\cdot\nabla\psi_\mathrm{a} = \delta(\boldsymbol{x})\delta(t) \tag{2.51}$$

with ψ_a the scalar acoustic potential. Now, since the source exists only for an infinitesimal instant of time $t = 0$, the scalar potential ψ_a will be zero for $t < 0$. Due to the radially symmetry, we may rewrite (2.51) in cylindrical coordinates for $r = |\boldsymbol{x}| > 0$ by

$$\frac{1}{c_0^2}\frac{\partial^2 \psi_\mathrm{a}}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi_\mathrm{a} = 0 \quad \text{for } r > 0\,. \tag{2.52}$$

According to Sec. 2.3 (see (2.24)) the solution is

$$\psi_\mathrm{a} = \frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\,. \tag{2.53}$$

The first term represents a spherically symmetric wave propagating in the direction of increasing values of r (outgoing wave) and the second term describes an incoming wave. Physically, we have to set g to zero, since according to *causality* (also known as the radiation condition) sound produced by a source must radiate away from this source.

To complete the solution, we have to determine the function f, which results in (see [5])

$$f(t - r/c_0) = \frac{1}{4\pi}\,\delta(t - r/c_0) \tag{2.54}$$

and the solution becomes

$$\psi_{\mathrm{a}}(\boldsymbol{x}, t) = \frac{1}{4\pi r}\,\delta(t - r/c_0) = \frac{1}{4\pi|\boldsymbol{x}|}\,\delta(t - |\boldsymbol{x}|/c_0)\,. \tag{2.55}$$

This represents a spherical pulse that is nonzero only on the surface of the sphere with $r = c_0 t > 0$, whose radius increases with the speed of sound c_0. It clearly vanishes everywhere for $t < 0$. Compared to the solution of a potential flow generated by a pulsating sphere (see Sec. 1.5) we have as an argument the retarded time.

2.6. Free space Green's functions

The *free-space Green's function* $G(\boldsymbol{x}, \boldsymbol{y}, t - \tau)$ is the *causal* solution of the wave equation by an impulsive point source with strength $\delta(\boldsymbol{x} - \boldsymbol{y})\delta(t - \tau)$ located at $\boldsymbol{x} = \boldsymbol{y}$ at time $t = \tau$. The expression for G is simply obtained from (2.55), when we replace the source position $\boldsymbol{x} = 0$ at time $t = 0$ by $\boldsymbol{x} - \boldsymbol{y}$ at $t - \tau$. This substitutions result in

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla \cdot \nabla\right) G = \delta(\boldsymbol{x} - \boldsymbol{y})\delta(t - \tau) \quad \text{where } G = 0 \text{ for } t < \tau \tag{2.56}$$

with

$$G(\boldsymbol{x}, \boldsymbol{y}, t) = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\,\delta\left(t - \tau - \frac{|\boldsymbol{x} - \boldsymbol{y}|}{c_0}\right)\,. \tag{2.57}$$

This describes an impulsive, spherical symmetric wave expanding from the source at $\boldsymbol{y}$ (therefore $\boldsymbol{y}$ are called the source coordinates) with the speed of sound c_0. The wave amplitude decreases inversely with the distance to the observation point $\boldsymbol{x}$.

Now, Green's function is the fundamental building block for the computation of the inhomogeneous wave equation with any generalized source distribution $\mathcal{F}(\boldsymbol{x}, t)$

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla \cdot \nabla\right) p^{\mathrm{a}} = \mathcal{F}(\boldsymbol{x}, t)\,. \tag{2.58}$$

The key idea is that the source distribution is regarded as a distribution of impulsive point sources

$$\mathcal{F}(\boldsymbol{x}, t) = \int_0^T \int_{-\infty}^{\infty} \mathcal{F}(\boldsymbol{y}, \tau)\delta(\boldsymbol{x} - \boldsymbol{y})\delta(t - \tau)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\tau\,.$$

Therefore, the outgoing wave solution for each constituent source strength

$$\mathcal{F}(\boldsymbol{y}, \tau)\delta(\boldsymbol{x} - \boldsymbol{y})\delta(t - \tau)$$

is given by

$$\mathcal{F}(\boldsymbol{y}, \tau)G(\boldsymbol{x}, \boldsymbol{y}, t)\,.$$

Therefore, the overall solution is obtained by adding up all the individual contributions

$$\begin{aligned} p_{\rm a}(\boldsymbol{x},t) &= \int\limits_0^T \int\limits_{-\infty}^{\infty} \mathcal{F}(\boldsymbol{y},\tau) G(\boldsymbol{x},\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\tau \\ &= \frac{1}{4\pi} \int\limits_0^T \int\limits_{-\infty}^{\infty} \frac{\mathcal{F}(\boldsymbol{y},\tau)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)\mathrm{d}\boldsymbol{y}\,\mathrm{d}\tau \\ &= \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} \frac{\mathcal{F}\left(\boldsymbol{y},t-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\,. \end{aligned} \tag{2.59}$$

This integral formula is called a **retarded** formula, since it represents the pressure at position $\boldsymbol{x}$ (observation point) and time t as a linear superposition of sources at $\boldsymbol{y}$ radiated at earlier times $t - |\boldsymbol{x}-\boldsymbol{y}|/c_0$. Thereby, the time of travel for the sound waves from the source point $\boldsymbol{y}$ to the observer point $\boldsymbol{x}$ is $|\boldsymbol{x}-\boldsymbol{y}|/c_0$.

In general, finding a (tailored) Green's function of given configuration (including, e.g., scatterer) is only marginally easier than the full solution of the inhomogeneous wave equation.Therefore, it is not possible to give a general recipe. However, it is important to note that often we can simplify a problem already by the corresponding integral formulation (as done above) using free field Green's function. Furthermore, the delta-function source may be rendered into a more easily treated from by spatial Fourier transform. Thereby, (2.57) leads to the free field Green's function in the frequency domain (setting $\tau = 0$)

$$\begin{aligned} \hat{G}(\boldsymbol{x},\omega) &= \int\limits_{-\infty}^{\infty} \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\delta\left(t-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right) e^{-j\omega t}\,\mathrm{d}t \\ &= \frac{e^{-jkr}}{4\pi r} \end{aligned} \tag{2.60}$$

with $r = |\boldsymbol{x}-\boldsymbol{y}|$ and $k = \omega/c_0$.

2.7. Monopoles, dipoles and quadrupoles

A volume point source $q(t)\delta(\boldsymbol{x})$ as a model of a pulsating sphere (as considered in Sec. 1.5) is called a *monopole* point source. Now, we consider a compressible fluid and the corresponding wave equation

$$\left(\frac{1}{c_0}\frac{\partial^2}{\partial t^2} - \nabla\cdot\nabla\right)\psi_{\rm a} = q(t)\delta(\boldsymbol{x})\,.$$

The solution can be simply obtained by using (2.59), replacing $p^{\rm a}$ by $\psi_{\rm a}$ and setting $\mathcal{F}(\boldsymbol{y},\tau) = q(\tau)\delta(\boldsymbol{y})$

$$\psi_{\rm a}(\boldsymbol{x},t) = \frac{q(t-|\boldsymbol{x}|/c_0)}{4\pi|\boldsymbol{x}|} = \frac{q(t-r/c_0)}{4\pi r}\,. \tag{2.61}$$

This differs from the solution obtained within an incompressible fluid (see (1.63)) by the dependence on the retarded time $t - r/c_0$. Any change at the source is now communicated to a fluid element at distance r after an appropriate estimated delay r/c_0 required for sound to travel outward from the source.

In a next step, we will investigate in a *point dipole*. Then, a source on the right hand side of the wave equation (2.58) of the following type

$$\mathcal{F}(\boldsymbol{x},t) = \nabla \cdot (\boldsymbol{f}(t)\delta(\boldsymbol{x})) = \frac{\partial}{\partial x_j}\left(f_j(t)\delta(\boldsymbol{x})\right) \tag{2.62}$$

is called a *point dipole* located at the origin. The sound generated by such a source computes according to (2.59)

$$p_{\mathrm{a}}(\boldsymbol{x},t) = \frac{1}{4\pi}\int\limits_0^T\int\limits_{-\infty}^{\infty} \frac{\partial}{\partial y_j}\left(f_j(\tau)\delta(\boldsymbol{y})\right) \frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\tau\,. \tag{2.63}$$

In a first step, we perform an integration by parts and arrive at

$$\begin{aligned} p_{\mathrm{a}}(\boldsymbol{x},t) &= -\frac{1}{4\pi}\int\limits_0^T\int\limits_{-\infty}^{\infty} f_j(\tau)\delta(\boldsymbol{y})\frac{\partial}{\partial y_j}\left(\frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\mathrm{d}\boldsymbol{y}\,\mathrm{d}\tau \\ &\quad +\frac{1}{4\pi}\int\limits_0^T\int\limits_{\Gamma} \left(f_j(\tau)\delta(\boldsymbol{y})\right)\frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\boldsymbol{n}\cdot\boldsymbol{e}_j\,\mathrm{d}s\,\mathrm{d}\tau\,. \end{aligned} \tag{2.64}$$

Thereby the second integral has to be evaluated at a surface for which $y_j = \pm\infty$, so that due to the property of the delta function $\delta(\boldsymbol{y}) = 0$ at $y_j = \pm\infty$, it vanishes. Furthermore, we explore the relation

$$\frac{\partial}{\partial y_j}\left(\frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\right) = -\frac{\partial}{\partial x_j}\left(\frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\right),$$

and arrive at

$$\begin{aligned} p_{\mathrm{a}}(\boldsymbol{x},t) &= \frac{1}{4\pi}\int\limits_0^T\int\limits_{-\infty}^{\infty} f_j(\tau)\delta(\boldsymbol{y})\frac{\partial}{\partial x_j}\left(\frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\mathrm{d}\boldsymbol{y}\,\mathrm{d}\tau \\ &= \frac{1}{4\pi}\frac{\partial}{\partial x_j}\int\limits_0^T\int\limits_{-\infty}^{\infty} f_j(\tau)\delta(\boldsymbol{y})\frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\tau\,. \end{aligned} \tag{2.65}$$

Due to the property of the delta function, we can directly obtain the solution for the acoustic pressure by

$$p_{\mathrm{a}}(\boldsymbol{x},t) = \frac{\partial}{\partial x_j}\left(\frac{f_j(t-|\boldsymbol{x}|/c_0)}{4\pi|\boldsymbol{x}|}\right). \tag{2.66}$$

Therefore, a distributed dipole source $\mathcal{F}(\boldsymbol{x},t) = \nabla\cdot\boldsymbol{f}(\boldsymbol{x},t)$ results in the following expression for the acoustic pressure

$$p_{\mathrm{a}}(\boldsymbol{x},t) = \frac{1}{4\pi}\frac{\partial}{\partial x_j}\int\limits_{-\infty}^{\infty}\frac{f_j(\boldsymbol{y},t-|\boldsymbol{x}-\boldsymbol{y}|/c_0)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\,. \tag{2.67}$$

A point dipole at the origin oriented in the direction of the unit vector $\boldsymbol{n}$ is entirely equivalent to two point monopoles of equal but opposite strengths placed a short distance apart (much smaller as the wavelength). Furthermore, a combination of four monopole sources, whose net volume source strength is zero, is called a *quadrupole*. A general quadrupole is a source distribution being characterized by a second space derivative of the form

$$\mathcal{F}(\boldsymbol{x},t) = \frac{\partial^2 L_{ij}}{\partial x_i \partial x_j} \,. \tag{2.68}$$

Here, L_{ij} are the components of an arbitrary tensor. In the context of aeroacoustics, $[\boldsymbol{L}]$ will denote the Lighthill tensor (see Sec. 3.1). Applying the procedure as in the case of the dipole source two times results in the corresponding acoustic pressure

$$p_{\mathrm{a}}(\boldsymbol{x},t) = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{\infty} \frac{L_{ij}(\boldsymbol{y}, t - |\boldsymbol{x}-\boldsymbol{y}|/c_0)}{|\boldsymbol{x}-\boldsymbol{y}|} \, \mathrm{d}\boldsymbol{y} \,. \tag{2.69}$$

2.8. Calculation of acoustic far field

We will now discuss useful approximations for the evaluation of the retarded potential formulation

$$p_{\mathrm{a}}(\boldsymbol{x},t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\mathcal{F}\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}\right)}{|\boldsymbol{x}-\boldsymbol{y}|} \, \mathrm{d}\boldsymbol{y} \,, \tag{2.70}$$

when computing the sound in the far field. Thereby, as mostly true for practical applications, we assume that $\mathcal{F}(\boldsymbol{x},t)$ is nonzero only in a finite source region, as displayed in Fig. 2.3. Furthermore, the source region contains the origin O of the coordinate system. In a first step,

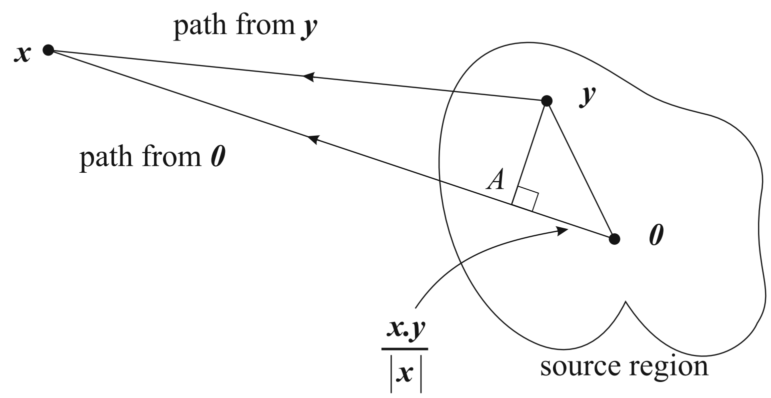


Figure 2.3.: Acoustic far field calculation.

we assume $|\boldsymbol{x}| \gg |\boldsymbol{y}|$, so that the following approximation will hold

$$\begin{aligned} |\boldsymbol{x}-\boldsymbol{y}| &= \left(|\boldsymbol{x}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} + |\boldsymbol{y}|^2\right)^{\frac{1}{2}} \\ &= |\boldsymbol{x}| \left(1 - \frac{2\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2} + \frac{|\boldsymbol{y}|^2}{|\boldsymbol{x}|^2}\right)^{\frac{1}{2}} \\ &\approx |\boldsymbol{x}| \left(1 - \frac{2\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2}\right)^{\frac{1}{2}} \\ &\approx |\boldsymbol{x}| - \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|} \quad \text{for} \quad \frac{|\boldsymbol{y}|}{|\boldsymbol{x}|} \ll 1\,. \end{aligned} \tag{2.71}$$

In a second step, we investigate in the term $1/|\boldsymbol{x}-\boldsymbol{y}|$ using the above result

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \approx \frac{1}{|\boldsymbol{x}| - \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|}} = \frac{1}{|\boldsymbol{x}|}\left(\frac{1}{1-\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2}}\right) \tag{2.72}$$

Now, we develop the term in the parenthesis in a Taylor series up to first order and arrive at

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \approx \frac{1}{|\boldsymbol{x}|}\left(1+\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2}\right) = \frac{1}{|\boldsymbol{x}|} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^3}\,.$$

This approximation demonstrates that in order to obtain the far field approximation of (2.70), which solution behaves like $1/r = 1/|\boldsymbol{x}|$ as $|\boldsymbol{x}| \to \infty$, it is sufficient to replace $|\boldsymbol{x}-\boldsymbol{y}|$ in the denominator of the integrand by $|\boldsymbol{x}|$. However, in the argument of the source strength $\mathcal{F}$ it is important to retain possible phase differences between the sound waves generated by the source distribution at location $\boldsymbol{y}$. Therefore, we replace $|\boldsymbol{x}-\boldsymbol{y}|$ in the source argument by the approximation obtained in (2.71) and arrive at

$$p_{\mathrm{a}}(\boldsymbol{x},t) \approx \frac{1}{4\pi|\boldsymbol{x}|}\int\limits_{-\infty}^{\infty} \mathcal{F}\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right)\,\mathrm{d}\boldsymbol{y}\,, \quad |\boldsymbol{x}| \to \infty\,. \tag{2.73}$$

This approximation when computing the acoustic far field is known as *Fraunhofer approximation*. The source region may extend over many characteristic wavelengths of the sound. By retaining the contribution $\boldsymbol{x}\cdot\boldsymbol{y}/(c_0|\boldsymbol{x}|)$ to the retarded time, we ensure that the interference between waves generated at different positions within the source region is correctly described by this far-field approximation. Let's consider the setup as displayed in Fig. 2.3. The acoustic travel time from a source point $\boldsymbol{y}$ to a far-field point $\boldsymbol{x}$ is equal to that from the point labeled by A to $\boldsymbol{x}$ when $\boldsymbol{x}$ goes to infinity. The travel time over the distance OA computes by

$$t_{OA} = \frac{1}{c_0}\,\boldsymbol{y}\cdot\boldsymbol{e}_{\boldsymbol{x}} = \frac{1}{c_0}\,\boldsymbol{y}\cdot\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\,.$$

Therefore, the time obtained by $|\boldsymbol{x}|/c_0 - \boldsymbol{x}\cdot\boldsymbol{y}/(c_0|\boldsymbol{x}|)$ is the correct value of the retarded time when $\boldsymbol{x}$ goes to infinity.

Let's apply the above approximation for a dipole source distribution. In doing so, we use the far-field formula according to (2.73) to a dipole source $\mathcal{F}(\boldsymbol{x},t) = \nabla \cdot \boldsymbol{f}(\boldsymbol{x},t)$ and obtain

$$\begin{aligned} p_{\mathrm{a}}(\boldsymbol{x},t) &\approx \frac{1}{4\pi}\frac{\partial}{\partial x_j}\left(\frac{1}{|\boldsymbol{x}|}\int\limits_{-\infty}^{\infty} f_j\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right)\,\mathrm{d}\boldsymbol{y}\right) \\ &= \frac{1}{4\pi|\boldsymbol{x}|}\frac{\partial}{\partial x_j}\left(\int\limits_{-\infty}^{\infty} f_j\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right)\,\mathrm{d}\boldsymbol{y}\right) \\ &\quad -\frac{1}{4\pi}\frac{x_j}{|\boldsymbol{x}|^3}\left(\int\limits_{-\infty}^{\infty} f_j\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right)\,\mathrm{d}\boldsymbol{y}\right) \end{aligned} \tag{2.74}$$

Since the second term decreases at least with a factor of $1/r^2$, we can neglect it for our far-field approximation. In a last step we will replace the space derivative with a time derivative, which is usually more easily estimated in practical applications. This operation is done as follows

$$\frac{\partial}{\partial x_j} f_j\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right) = \frac{\partial f_j}{\partial t}\frac{\partial}{\partial x_j}\left(t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right).$$

Now, the second term evaluates as

$$\begin{aligned} \frac{\partial}{\partial x_j}\left(t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right) &= -\frac{1}{c_0}\frac{\partial|\boldsymbol{x}|}{\partial x_j} + \frac{1}{c_0}\frac{\partial}{\partial x_j}\left(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|}\right) \\ &= -\frac{1}{c_0}\frac{x_j}{|\boldsymbol{x}|} + \frac{1}{c_0}\frac{y_j|\boldsymbol{x}| - \boldsymbol{x}\cdot\boldsymbol{y}\,x_j|\boldsymbol{x}|^{-1}}{|\boldsymbol{x}|^2} \\ &= -\frac{1}{c_0}\frac{x_j}{|\boldsymbol{x}|} + \frac{y_j}{c_0|\boldsymbol{x}|} - \frac{\boldsymbol{x}\cdot\boldsymbol{y}\,x_j}{|\boldsymbol{x}|^3} \\ &\approx -\frac{1}{c_0}\frac{x_j}{|\boldsymbol{x}|} \quad \text{for } |\boldsymbol{y}| \ll |\boldsymbol{x}|\,. \end{aligned}$$

Collecting these results, we can provide the far-field approximation for a source dipole $\mathcal{F} = \nabla\cdot \boldsymbol{f}(\boldsymbol{x},t)$ as follows (canceling all terms which are proportional to $1/|\boldsymbol{x}|^2$ as well as $\boldsymbol{x}\cdot\boldsymbol{y}\,x_j/|\boldsymbol{x}|^4$)

$$p_{\mathrm{a}}(\boldsymbol{x},t) \approx \frac{-x_j}{4\pi c_0|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\left(\int\limits_{-\infty}^{\infty} f_j\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right)\,\mathrm{d}\boldsymbol{y}\right) \tag{2.75}$$

Please note that the term

$$\frac{x_j}{|\boldsymbol{x}|^2} = \frac{x_j}{|\boldsymbol{x}|}\frac{1}{|\boldsymbol{x}|} = \frac{x_j}{|\boldsymbol{x}|}\frac{1}{r}$$

is not changing the rate of the amplitude decay, which is still given by $1/r$. The first term $x_j/|\boldsymbol{x}|$ is the jth component of the unit vector $\boldsymbol{x}/|\boldsymbol{x}|$ and so it does just influence the directivity pattern (see Fig. 2.4 for the directivity of a dipole).

Furthermore, it is necessary to realize that the rule of interchanging a space derivative with a time derivative is given by

$$\frac{\partial}{\partial x_j} \approx -\frac{1}{c_0}\frac{x_j}{|\boldsymbol{x}|}\frac{\partial}{\partial t}\,. \tag{2.76}$$

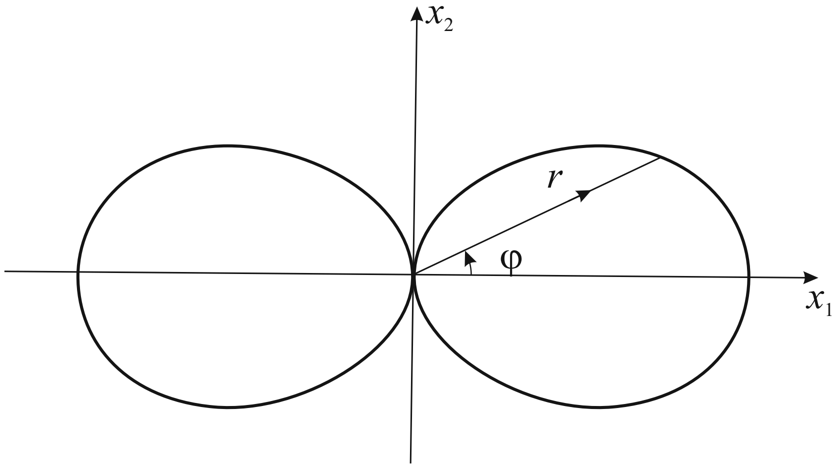


Figure 2.4.: Directivity of a dipole source. The plotted directivity is $\propto p^2$ (proportional to the intensity).

We will now explore this relation, when deriving the far-field approximation for a quadrupole source given by

$$\mathcal{F}(\boldsymbol{x},t) = \frac{\partial^2 L_{ij}(\boldsymbol{x},t)}{\partial x_i \partial x_j}\,. \tag{2.77}$$

According to (2.76) we directly arrive at

$$p_{\mathrm{a}}(\boldsymbol{x},t) \approx \frac{x_i x_j}{4\pi c_0 |\boldsymbol{x}|^3}\frac{\partial^2}{\partial t^2}\left(\int\limits_{-\infty}^{\infty} L_{ij}\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right)\,\mathrm{d}\boldsymbol{y}\right) \tag{2.78}$$

Now, for a point quadrupole in the x_1-x_2 plane

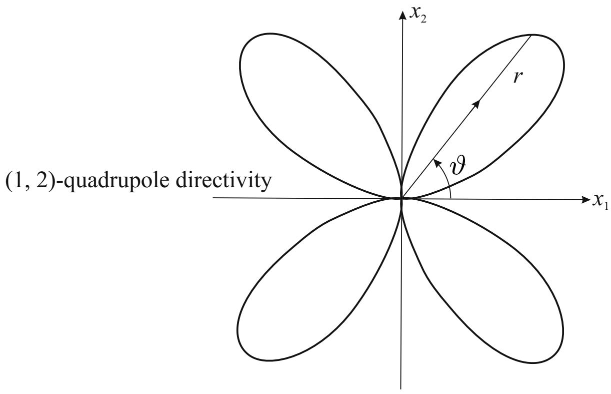


Figure 2.5.: Directivity of a quadrupole source: radiation in the x_1-x_2 plane ($\varphi = 0, \pi$).

$$\mathcal{F}(\boldsymbol{x},t) = \frac{\partial^2}{\partial x_1 \partial x_2}\,\left(L(t)\delta(\boldsymbol{x})\right)$$

we obtain by exploring (2.78) the following far-field pressure

$$p_{\mathrm{a}}(\boldsymbol{x},t) \approx \frac{x_1 x_2}{4\pi c_0 |\boldsymbol{x}|^3}\frac{\partial^2}{\partial t^2} L\left(t - \frac{|\boldsymbol{x}|}{c_0}\right) \qquad \boldsymbol{x}\to\infty\,. \tag{2.79}$$

If we use spherical coordinates, such that

$$x_1 = r\cos\vartheta\,; \quad x_2 = r\sin\vartheta\cos\varphi\,; \quad x_3 = r\sin\vartheta\cos\varphi$$

we may rewrite the pressure by

$$p_\mathrm{a}(\boldsymbol{x},t) \approx \frac{\sin 2\vartheta\ \cos\varphi}{8\pi c_0|\boldsymbol{x}|}\frac{\partial^2}{\partial t^2}L\left(t-\frac{|\boldsymbol{x}|}{c_0}\right) \qquad \boldsymbol{x}\to\infty\,. \tag{2.80}$$

The directivity pattern of the sound intensity, which is given by $\propto (p^\mathrm{a})^2$, is therefore represented by $\sin^2 2\vartheta\ \cos^2\varphi$. Its shape is displayed in Fig. 2.5.

2.9. Questions: Chapter 2

1. Derive conservation equations for linear acoustics (isentropic case) with a background flow by using the following perturbation ansatz

$$p_\mathrm{a} = p - p_0\,; \quad \rho_\mathrm{a} = \rho-\rho_0\,; \boldsymbol{v}_\mathrm{a} = \boldsymbol{v}-\boldsymbol{v}_0\,.$$

   Assume that the mean quantities p_0, ρ_0 do not depend on space and time and that the mean velocity $\boldsymbol{v}_0$ does not depend on time.

2. Apply the perturbation ansatz

$$p(t) = p_\mathrm{a}(t)+p_0\,; \quad \rho(t) = \rho_\mathrm{a}(t)+\rho_0\,; \quad \boldsymbol{v}(t) = \boldsymbol{v}_\mathrm{a}(t)$$

   to (1.47) and determine by neglecting second order terms the relation between p_a and ρ_a. Under which consitions you obtain

$$p_\mathrm{a} = c^2\rho_\mathrm{a}\,.$$

3. Provide the free space Green's functions and discuss its property!

4. Explain the idea how the acoustic pressure at an observer point $\boldsymbol{x}$ can be computed for any source distribution.

5. How does the source distributions look like for a monopole, dipole as well as quadrupole?

6. Explain the Fraunhofer approximation for the computation of the acoustic pressure!

7. Show that for the far field the following approximation is valid

$$\frac{\partial}{\partial x_j}f_j\left(\boldsymbol{y},t-\frac{|\boldsymbol{x}|}{c_0}+\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right) \approx \frac{1}{c_0}\frac{x_j}{|\boldsymbol{x}|}\frac{\partial}{\partial t}f_j\left(\boldsymbol{y},t-\frac{|\boldsymbol{x}|}{c_0}+\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{c_0|\boldsymbol{x}|}\right)$$

8. Assume a plane wave in air at ambient temperature with an acoustic SPL of 90 dB. Compute the acoustic density, the acoustic particle velocity, the acoustic intensity and the acoustic power (for a surface of $1\,\mathrm{m}^2$). Provide the results both in their physical units as well as in decibel (dB).

9. We assume a breathing sphere of radius $R = 0.1\,\mathrm{m}$. At a distance of $5\,\mathrm{m}$, an SPL of $80\,\mathrm{dB}$ at $1\,\mathrm{kHz}$ is measured. Compute the acoustic intensity as well as the total acoustic power radiated by the vibrating sphere.

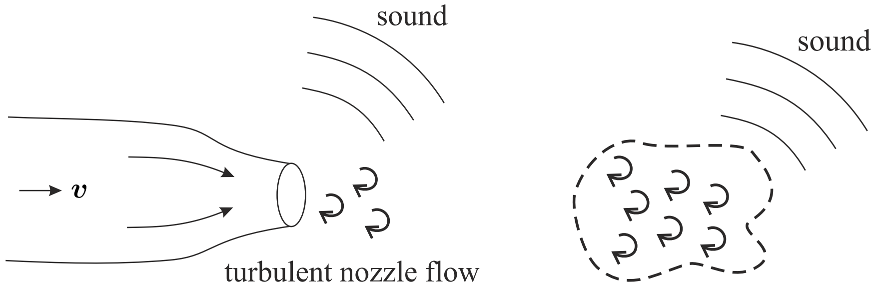
10. Which conditions have to be fulfilled for a region to be acoustically compact? For such an case, the acoustic wave equation can be simplified to which one? Explain the procedure.

3. Aeroacoustics

3.1. Lighthill's Acoustic Analogy

The sound generated by a flow in an unbounded fluid is usually called *aerodynamic sound*. Most unsteady flows in technical applications are of high Reynolds number, and the acoustic radiation is a very small by-product of the motion. Thereby, the turbulence is usually produced by fluid motion over a solid body and/or by flow instabilities. Lighthill transformed the general equations of mass and momentum conservation to an exact inhomogeneous wave equation whose source terms are important only within the turbulent region [6, 7].

Lighthill was initially interested in solving the problem, illustrated in Fig. 3.1a, of the sound produced by a turbulent nozzle and arrived at the inhomogeneous wave equation. However, at this time a volume discretization by numerical schemes was not feasible and so a



(a) Turbulent nozzle flow. (b) Isolated turbulent region.

Figure 3.1.: Sound generation by turbulent flows.

transformation of the PDE into an integral representation was performed, which can just be achieved for a free field setup, for which Green's function is available. Therefore, Lighthill's theory in its integral formulation just applies to the simple situation as given in Fig. 3.1b. This avoids complications caused by the presence of the nozzle. The fluid is assumed to be at rest at the observer position, where a mean pressure, density and speed of sound are respectively equal to p_0, ρ_0 and c_0. So Lighthill compared the equations for the production of density fluctuations in the real flow with those in an ideal linear acoustic medium (quiescent fluid).

For the derivation, we start at Reynolds form of the momentum equation, as given by (1.18) neglecting any force density $\boldsymbol{f}$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot [\boldsymbol{\pi}] \;=\; 0\,, \tag{3.1}$$

with the momentum flux tensor $\pi_{ij} = \rho v_i v_j + (p - p_0)\delta_{ij} - \tau_{ij}$, where the constant pressure p_0 is inserted for convenience. In an ideal, linear acoustic medium, the momentum flux tensor contains only the pressure

$$\pi_{ij} \to \pi^0_{ij} = (p - p_0)\delta_{ij} = c_0^2(\rho - \rho_0)\delta_{ij} \tag{3.2}$$

and Reynolds momentum equation reduces to

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_i}\left(c_0^2(\rho - \rho_0)\right) = 0\,. \tag{3.3}$$

Rewriting the conservation of mass in the form

$$\frac{\partial}{\partial t}(\rho - \rho_0) + \frac{\partial \rho v_i}{\partial x_i} = 0 \tag{3.4}$$

allows us to eliminate the momentum density ρv_i in (3.3). Therefore, we perform a time derivative on (3.4), a spatial derivative on (3.3) and substract the two resulting equations. These operations leads to the equation of linear acoustics satisfied by the perturbation density

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla\cdot\nabla\right)\left(c_0^2(\rho - \rho_0)\right) = 0\,. \tag{3.5}$$

Because flow is neglected, the unique solution of this equation satisfying the radiation condition and we obtain $\rho - \rho_0 = 0$.

Now, it can be asserted that the sound generated by the turbulence in the *real fluid* is exactly equivalent to that produced in the ideal, stationary acoustic medium forced by the stress distribution

$$L_{ij} = \pi_{ij} - \pi_{ij}^0 = \rho v_i v_j + \left((p - p_0) - c_0^2(\rho - \rho_0)\right)\delta_{ij} - \tau_{ij}\,, \tag{3.6}$$

where $[\boldsymbol{L}]$ is called the *Lighthill stress tensor*.

Indeed, we can rewrite (3.1) as the momentum equation for an ideal, stationary acoustic medium of mean density ρ_0 and speed of sound c_0 subjected to the externally applied stress L_{ij}

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \pi_{ij}^0}{\partial x_j} \quad = \quad -\frac{\partial}{\partial x_j}\left(\pi_{ij} - \pi_{ij}^0\right)\,, \tag{3.7}$$

or equivalent

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_j}\left(c_0^2(\rho - \rho_0)\right) \quad = \quad -\frac{\partial L_{ij}}{\partial x_j}\,. \tag{3.8}$$

By eliminating the momentum density ρv_i using (3.4) we arrive at **Lighthill's equation**

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla\cdot\nabla\right)\left(c_0^2(\rho - \rho_0)\right) = \frac{\partial^2 L_{ij}}{\partial x_i \partial x_j}\,. \tag{3.9}$$

It has to be noted that $(\rho - \rho_0) = \rho'$ is a fluctuating density not being equal to the acoustic density ρ^{a}, but a superposition of flow and acoustic parts within flow regions.

The problem of calculating the flow generated sound is equivalent to solving this wave equation, which is possible when the source term $\partial^2 L_{ij}/\partial x_i \partial x_j$ is provided, e.g., by a CFD computation. This type of source term can be interpreted as a quadrupole term. Therefore, the free field turbulence is an extremely weak sound source, and so in low Mach number flows just a very small portion of the flow energy is converted into sound. However, in the presence of walls the sound radiation by turbulence can be dramatically enhanced. In the next section, we will see that compact bodies will radiate a dipole sound field associated to the force which they exert on the flow as a reaction to the dynamic force of the flow applied to them. Sharp edges are particularly efficient radiators.

In the definition of the Lighthill tensor according to (3.6) the term $\rho v_i v_j$ is called the Reynolds stress. It is a nonlinear term and can be neglected except where the motion is turbulent. The second term $\big((p-p_0)-c_0^2(\rho-\rho_0)\big)\,\delta_{ij}$ represents the *excess of* moment transfer by the pressure over that in the ideal fluid of density ρ_0 and speed of sound c_0. This is produced by wave amplitude nonlinearity, and by mean density variations in the source flow. The viscous stress tensor τ_{ij} properly accounts for the attenuation of the sound. In most applications the Reynolds number in the source region is high and we can neglect this contribution.

The solution of (3.9) for free field radiation condition with outgoing wave behavior can be rewritten in integral form as follows (see sec. 2.7)

$$c_0^2(\rho-\rho_0)(\boldsymbol{x},t) = \frac{1}{4\pi}\,\frac{\partial^2}{\partial x_i \partial x_j} \int\limits_{-\infty}^{\infty} \frac{L_{ij}(\boldsymbol{y}, t-|\boldsymbol{x}-\boldsymbol{y}|/c_0)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\,. \tag{3.10}$$

Thereby, $\boldsymbol{y}$ defines the source coordinate and $\boldsymbol{x}$ the coordinate at which we compute the acoustic density fluctuation. This provides a useful prediction of the sound, if L_{ij} is known. Please note that the terms in L_{ij} not only account for the generation of sound, but also includes acoustic *self modulation* caused by

- acoustic nonlinearity,
- the convection of sound waves by the turbulent flow velocity,
- refraction caused by sound speed variations,
- and attenuation due to thermal and viscous actions.

The influence of acoustic nonlinearity and thermoviscous dissipation is usually sufficiently small to be neglected within the source region. Convection and refraction of sound within the flow region can be important, e.g., in the presence of a mean shear layer (when the Reynolds stress will include terms like $\rho v_{0i} v_j'$, where $\boldsymbol{v}_0$ and $\boldsymbol{v}'$ respectively denote the mean and fluctuating components of $\boldsymbol{v}$). Such effects are described by the presence of unsteady linear terms in L_{ij}. Furthermore, since for practical applications the source term is obtained by numerically solving Navier-Stokes equation, the question of how accurate the source term is resolved, is always present.

Now, let's consider the situation for which the mean density and speed of sound are uniform throughout the fluid. The variations in the density ρ within a low Mach number, high Reynolds number source flow are then of order $O(\rho_0 \mathrm{Ma}^2)$. Thus, $\rho v_i v_j = \rho_0\left(1+O(\mathrm{Ma}^2)\right) v_i v_j \approx \rho_0 v_i v_j$. Furthermore, if $c(\boldsymbol{x},t)$ is the local speed of sound in the source region, it can be shown that $c_0^2/c^2 = 1 + O(\mathrm{Ma}^2)$, so that we obtain

$$p - p_0 - c_0^2(\rho-\rho_0) = (p-p_0)(1 - c_0^2 \underbrace{\frac{\rho-\rho_0}{p-p_0}}_{1/c^2}) \approx (p-p_0)(1-c_0^2/c^2) \sim O(\rho_0 v^2 \mathrm{Ma}^2)\,. \tag{3.11}$$

Therefore, if viscous dissipation is neglected, we may approximate the Lighthill tensor by

$$L_{ij} \approx \rho_0 v_i v_j \quad \text{for } \mathrm{Ma}^2 \ll 1\,. \tag{3.12}$$

Please note that with this assumptions, the divergence of (1.18) provides the following equivalence (assuming an incompressible flow $\nabla \cdot \boldsymbol{v} = 0$ and $\boldsymbol{f} = 0$)

$$\nabla \cdot \nabla p^{\mathrm{ic}} = -\rho_0 \frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} \tag{3.13}$$

with the incompressible flow pressure p^{ic}. With this result we obtain for the pressure fluctuation using the isentropic pressure-density relation the following integral representation

$$p'(\boldsymbol{x}, t) \approx \frac{\partial^2}{\partial x_i \partial x_j} \int \frac{\rho_0 v_i v_j (\boldsymbol{y}, t - |\boldsymbol{x} - \boldsymbol{y}|/c_0)}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \, \mathrm{d}\boldsymbol{y} \tag{3.14}$$

$$\approx \frac{x_i x_j}{4\pi c_0^2 |\boldsymbol{x}|^3} \frac{\partial^2}{\partial t^2} \int \rho_0 v_i v_j \left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}|}{c_0} + \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{c_0 |\boldsymbol{x}|} \right) \mathrm{d}\boldsymbol{y}\,. \tag{3.15}$$

To obtain (3.15), we have used the far field approximation, which allows the following substitution (see Sec. 2.8)

$$\frac{\partial}{\partial x_j} \approx -\frac{1}{c_0} \frac{x_j}{|\boldsymbol{x}|} \frac{\partial}{\partial t}\,. \tag{3.16}$$

Now, we want to derive the order of the magnitude of the acoustic pressure as a function of the flow velocity $\boldsymbol{v}$. In doing so, we introduce a characteristic velocity v and length scale l of a single vortex as displayed in Fig. 3.2. Fluctuations in $v_i v_j$ occurring in different turbulent

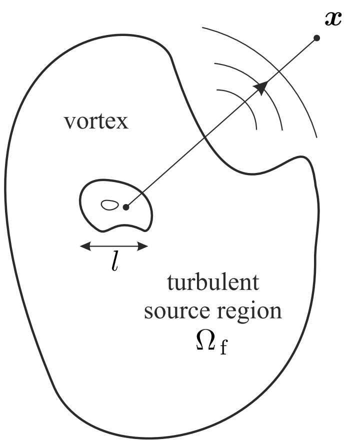


Figure 3.2.: Single vortex in a turbulent flow region at its acoustic radiation towards the far field.

regions by distances larger than $O(l)$ will be treated to be statistically independent. So the sound may be considered to be generated by a collection of Ω_{f}/l^3 independent vortices. The characteristic frequency of the turbulent fluctuations can be estimated by $f \sim v/l$ so that the wavelength λ of sound will result in

$$\lambda = \frac{c_0}{f} \sim \frac{c_0 l}{v} = \frac{l}{\mathrm{Ma}} \gg l \text{ for } \mathrm{Ma} = v/c_0 \ll 1\,.$$

Hence, we arrive at the quite important conclusion that the turbulent vortices are all acoustically compact. This means that in the relation (3.15) the retarded time variation $\boldsymbol{x}\cdot\boldsymbol{y}/(c_0|\boldsymbol{x}|)$ can be neglected. Therefore, the value of the integral over one source vortex in (3.15) can be estimated to be of order $\rho_0 v^2 l^3$. The order of the magnitude for the time derivative in (3.15) is estimated to be

$$\frac{\partial}{\partial t} \sim \frac{v}{l}\,.$$

Collecting all this estimates, we may now state that the acoustic pressure in the far-field, generated by one vortex, satisfies

$$p_{\mathrm{a}} \sim \frac{l}{|\boldsymbol{x}|}\,\frac{\rho_0 v^4}{c_0^2} = \frac{l}{|\boldsymbol{x}|}\,\rho_0 v^2 \mathrm{Ma}^2\,. \tag{3.17}$$

The acoustic power defined by

$$P_{\mathrm{a}} = \oint\limits_{\Gamma} p_{\mathrm{a}} \boldsymbol{v}_{\mathrm{a}} \cdot \mathrm{d}\boldsymbol{s} = \oint\limits_{\Gamma} p_{\mathrm{a}} \boldsymbol{v}_{\mathrm{a}} \cdot \boldsymbol{n}\;\mathrm{d}s \tag{3.18}$$

can be computed in the far-field with the relation $\boldsymbol{v}_{\mathrm{a}} \cdot \boldsymbol{n} = p_{\mathrm{a}}/(\rho_0 c_0)$ as follows

$$P_{\mathrm{a}} = \oint\limits_{\Gamma} \frac{{p_{\mathrm{a}}}^2}{\rho_0 c_0}\;\mathrm{d}s\,. \tag{3.19}$$

This formula allows us to estimate the acoustic power generated by one vortex

$$P_{\mathrm{a}} \sim 4\pi|\boldsymbol{x}|^2\,\frac{{p_{\mathrm{a}}}^2}{\rho_0 c_0} \sim \frac{l^2 \rho_0 v^8}{c_0^5} = \rho_0\, l^2\, v^3\, \mathrm{Ma}^5\,. \tag{3.20}$$

This is the famous **eighth power law**.

3.2. Curle's Theory

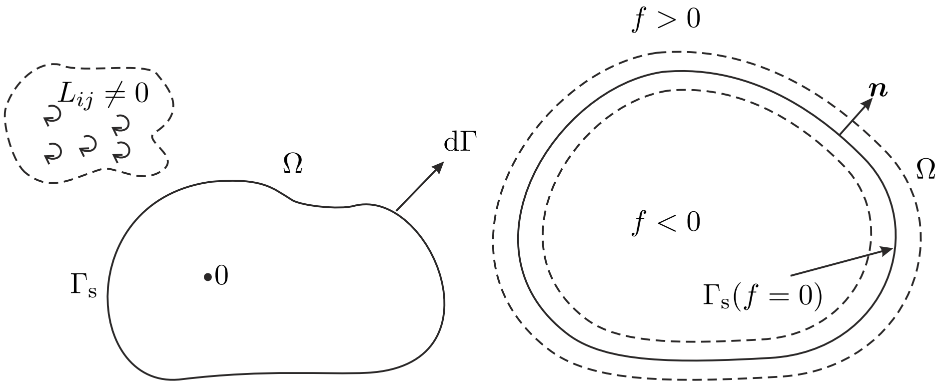
The main restriction of Lighthill's integral formulation is that it can just consider free radiation. Therewith, it can not consider situations where there is any solid body within the region. In [8] this problem was solved by deriving an integral formulation for the sound generated by turbulence in the vicinity of an arbitrary, fixed surface Γ_{s} as displayed in Fig. 3.3a. Thereby the surface Γ_{s} is defined by the function $f(\boldsymbol{x})$, which has the following property (see Fig. 3.3b).

$$f(\boldsymbol{x}) = \begin{cases} 0 & \text{for } \boldsymbol{x} \text{ on } \Gamma_{\mathrm{s}} \\ < 0 & \text{for } \boldsymbol{x} \text{ within the surface} \\ > 0 & \text{for } \boldsymbol{x} \text{ in } \Omega\,. \end{cases}$$

This surface may either be a solid body, or just an artificial control surface used to isolate a fixed region of space containing both solid bodies and fluid or just fluid.

To derive Curle's equation we start with the momentum equation according to (3.8) and multiply it with the Heaviside function $H(f)$

$$H(f)\,\frac{\partial \rho v_i}{\partial t} + H(f)\,\frac{\partial}{\partial x_i}\left(c_0^2(\rho-\rho_0)\right) \;=\; -H(f)\,\frac{\partial L_{ij}}{\partial x_j}\,. \tag{3.21}$$



(a) Turbulent source region with solid body. (b) Solid body.

Figure 3.3.: Setup for deriving Curle's equation.

Now, we use the product rule for differentiation (noting that the time derivative of the Heaviside function, which just depends on space, is zero) and obtain by also writing L_{ij} by its individual components

$$\frac{\partial}{\partial t}\left(\rho v_i H(f)\right) + \frac{\partial}{\partial x_j}\left(c_0^2(\rho - \rho_0)H(f)\right) - c_0^2(\rho-\rho_0)\frac{\partial H(f)}{\partial x_i} \tag{3.22}$$
$$= -\frac{\partial}{\partial x_j}\left(L_{ij}H(f)\right) + \underbrace{\left(\rho v_i v_j + \left((p-p_0) - c_0^2(\rho-\rho_0)\right)\delta_{ij} - \tau_{ij}\right)}_{L_{ij}}\frac{\partial H(f)}{\partial x_j}\,.$$

We can cancel out the term $c_0^2(\rho - \rho_0)\,\partial H(f)/\partial x_i$ being at both sides of the equation and arrive at

$$\frac{\partial}{\partial t}\left(\rho v_i H(f)\right) + \frac{\partial}{\partial x_j}\left(c_0^2(\rho-\rho_0)H(f)\right)$$
$$= -\frac{\partial}{\partial x_j}\left(L_{ij}H(f)\right) + \left(\rho v_i v_j + (p-p_0)\,\delta_{ij} - \tau_{ij}\right)\frac{\partial H(f)}{\partial x_j}\,. \tag{3.23}$$

The same procedure is now applied to the mass conservation according to (3.4) and so we obtain

$$\frac{\partial}{\partial t}\left((\rho-\rho_0)H(f)\right) + \frac{\partial}{\partial x_i}\left(\rho v_i H(f)\right) - \rho v_i\frac{\partial H(f)}{\partial x_i} = 0\,. \tag{3.24}$$

Now, we perform a time derivative to this equation and rearrange it for $\rho v_i H(f)$

$$\frac{\partial^2}{\partial t \partial x_i}\left(\rho v_i H(f)\right) = \frac{\partial}{\partial t}\left(\rho v_i \frac{\partial H(f)}{\partial x_i}\right) - \frac{\partial^2}{\partial t^2}\left((\rho-\rho_0)H(f)\right)\,. \tag{3.25}$$

In a last step, we apply the divergence operation to (3.23) and substitute the expression for

$\rho v_i H(f)$ from (3.25)

$$\begin{aligned}\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right. &- \left.\nabla\cdot\nabla\right)\left(c_0^2(\rho-\rho_0)H(f)\right) \qquad (3.26)\\ &= \frac{\partial^2 L_{ij}H(f)}{\partial x_i \partial x_j}\\ &- \frac{\partial}{\partial x_i}\left((\rho v_i v_j + (p-p_0)\delta_{ij} - \tau_{ij})\,\frac{\partial H(f)}{\partial x_j}\right)\\ &+ \frac{\partial}{\partial t}\left(\rho v_j \frac{\partial H(f)}{\partial x_j}\right).\end{aligned}$$

This equation is now valid throughout the space, including the region enclosed by Γ_s. Furthermore, compared to Lighthill's equation, we have obtained two additional terms on the right hand side of the wave equation including space derivatives of the Heaviside function $H(f)$. Thereby, according to our previous investigation the second term on the right hand side corresponds to a dipole and the third term to a monopole with the following interpretation:

- Γ_s is the boundary of a solid body:
  In this case the surface dipole represents the production of sound by the unsteady force that the body exerts on the exterior fluid, whereas the monopole is responsible for the sound generated by volume pulsations (if any) of the body.
- Γ_s is just an artificial control surface:
  The dipole and monopole sources account for the presence of solid bodies and turbulences within Γ_s (when L_{ij} is different from zero in Γ_s) and also for the interaction of sound generated outside Γ_s with the fluid and solid bodies inside Γ_s.

To transform (3.26) to the corresponding integral representation is a straight forward operation. According to the wave equation and its integral representation we obtain for the monopole term

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\langle\rho v_j\rangle\,\frac{\partial H(f)}{\partial y_j}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\,, \qquad (3.27)$$

where $\langle\ \rangle$ indicates that the term has to be evaluated at $(\boldsymbol{y}, t-|\boldsymbol{x}-\boldsymbol{y}|/c_0)$. For the dipole term we obtain the following integral representation

$$\frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{-\infty}^{\infty}\langle\rho v_i v_j + (p-p_0)\delta_{ij} - \tau_{ij}\rangle\,\frac{\partial H(f)}{\partial y_j}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\,. \qquad (3.28)$$

The last step is the handling of the Heaviside function, which has according to generalized function theory the following property for an arbitrary smooth function $\Phi(\boldsymbol{x})$ [5]

$$\int_{-\infty}^{\infty}\Phi(\boldsymbol{y})\frac{\partial H(f)}{\partial y_j}\,\mathrm{d}\boldsymbol{y} = \oint_{\Gamma_s}\Phi(\boldsymbol{y})n_j\,\mathrm{d}s = \oint_{\Gamma_s}\Phi(\boldsymbol{y})\,\mathrm{d}s_j\,. \qquad (3.29)$$

Exploring this property, we finally arrive at Curle's equation in integral form

$$\begin{aligned} c_0^2(\rho-\rho_0)\,H(f) &= \frac{\partial^2}{\partial x_i \partial x_j}\int\limits_\Omega \frac{\langle L_{ij}\rangle}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y} \\ & -\frac{\partial}{\partial x_i}\oint\limits_{\Gamma_\mathrm{s}} \frac{\langle \rho v_i v_j + (p-p_0)\delta_{ij} - \tau_{ij}\rangle}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}s_j(\boldsymbol{y}) \\ & +\frac{\partial}{\partial t}\oint\limits_{\Gamma_\mathrm{s}} \frac{\langle \rho v_j\rangle}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}s_j(\boldsymbol{y})\,. \end{aligned} \tag{3.30}$$

Now, let us restrict to a rigid body for which the flow velocity in normal direction on this body is zero, so that (3.30) reduces to

$$\begin{aligned} c_0^2(\rho-\rho_0)\,H(f) &= \frac{\partial^2}{\partial x_i \partial x_j}\int\limits_\Omega \frac{\langle L_{ij}\rangle}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y} \\ & -\frac{\partial}{\partial x_i}\oint\limits_{\Gamma_\mathrm{s}} \frac{\langle (p-p_0)\delta_{ij} - \tau_{ij}\rangle}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}s_j(\boldsymbol{y})\,. \end{aligned} \tag{3.31}$$

Furthermore, we assume the body to be acoustically compact, which means that $\mathrm{Ma} = v/c_0 \ll 1$. In the following investigation, we want to estimate the order of the sound generation by the dipole term. With the characteristic velocity v and scale length l of a vortex, we have the following relations

$$(p-p_0) \sim \rho_0 v^2\,; \quad \tau \sim \mu_\mathrm{f}\frac{v}{l}\,. \tag{3.32}$$

Therefore, we can compute the ratio

$$\frac{p-p_0}{\tau} \sim \frac{\rho_0 v l}{\mu_\mathrm{f}} = \frac{vl}{\nu_\mathrm{f}}$$

with $\nu_\mathrm{f} = \mu_\mathrm{f}/\rho_0$ the kinematic viscosity, which is just the definition of the Reynolds number $Re = vl/\nu_\mathrm{f}$. Since in turbulent flows Re is quite high, we can neglect the viscous contribution. In the far-field, the acoustic pressure p_a is equal to $c_0^2(\rho-\rho_0)H(f)$ ($H(f)$ is there just 1), and exploring the compactness which allows us to neglect the retarded time variation $\boldsymbol{x}\cdot\boldsymbol{y}/(c_0|\boldsymbol{x}|)$ results for the second term in (3.31) using (3.16) in

$$p_\mathrm{a} \approx \frac{x_i}{4\pi c_0|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\oint\limits_{\Gamma_\mathrm{s}}(p-p_0)\left(\boldsymbol{y}, t-\frac{|\boldsymbol{x}|}{c_0}\right)\mathrm{d}s_i = \frac{x_i}{4\pi c_0|\boldsymbol{x}|^2}\frac{\partial F_i}{\partial t}\left(t-\frac{|\boldsymbol{x}|}{c_0}\right) \tag{3.33}$$

with $\boldsymbol{F}$ the total unsteady surface force. For a surface element with a diameter of l, the contribution to the acoustic pressure p_a can be estimated by

$$\frac{1}{c_0|\boldsymbol{x}|}\,\frac{v}{l}\,\rho_0 v^2 l^2 = \frac{l}{|\boldsymbol{x}|}\,\rho_0 v^2 \mathrm{Ma}\,. \tag{3.34}$$

Assuming to have Γ_s/l^2 independently radiating surface elements, we can estimate the acoustic power (see also (3.20))

$$\begin{aligned} P_\mathrm{a} \sim 4\pi|\boldsymbol{x}|^2\frac{{p_\mathrm{a}}^2}{\rho_0 c_0}\frac{\Gamma_\mathrm{s}}{l^2} &= \left(4\pi|\boldsymbol{x}|^2\frac{l^2}{\rho_0 c_0|\boldsymbol{x}|^2}\,\rho_0^2 v^4 \mathrm{Ma}^2\right)\frac{\Gamma_\mathrm{s}}{l^2} \\ &\sim \rho_0\Gamma_\mathrm{s} v^3 \mathrm{Ma}^3\,. \end{aligned} \tag{3.35}$$

So, we see that in case of a dipole we arrive at a **sixth power law** and compared to the quadrupole we have a factor of $1/\mathrm{Ma}^2$ being stronger.

Therefore, we can summarize that Lighthills' inhomogeneous wave equation is a quite general model to describe flow-induced sound. Solving this partial differential equation by a volume discretization method includes all sources of the sound. The additional source terms, as given in (3.30) just come up, because the partial differential equation is converted to an integral representation for which Greens' function is needed. Furthermore, the solution is a fluctuating pressure (density), which approaches the acoustic pressure (density) outside the flow region.

3.3. Vortex Sound

Restricting to low Mach number flows and neglecting combustion and entropy sources of sound, we arrive at the classical theory of vortex sound [5]. For a real fluid, we may decompose the flow velocity $\boldsymbol{v}$ according to Helmholtz by

$$\boldsymbol{v} = \nabla \times \boldsymbol{\psi} + \nabla \phi = \boldsymbol{v}_{\mathrm{ic}} \ + \nabla \phi \,, \tag{3.36}$$

where $\boldsymbol{v}_{\mathrm{ic}}$ contains the solenoidal part of the flow velocity $\boldsymbol{v}$ and has the property $\nabla \cdot \boldsymbol{v}_{\mathrm{ic}} = 0$ (defines the incompressible part of $\boldsymbol{v}$). Furthermore, ϕ denotes the scalar velocity potential and $\boldsymbol{\psi}$ the vector potential.Thereby, the vorticity $\boldsymbol{\omega}$ is defined by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \nabla \times \boldsymbol{v}_{\mathrm{ic}} \,. \tag{3.37}$$

Knowing $\boldsymbol{v}$, we can compute the scalar potential by

$$\nabla \cdot \nabla \phi = \nabla \cdot \boldsymbol{v} - \nabla \cdot \nabla \times \boldsymbol{\psi} = \nabla \cdot \boldsymbol{v} \,, \tag{3.38}$$

and by taking the **curl**, we obtain for the vector potential

$$\nabla \times \nabla \times \boldsymbol{\psi} = \nabla \times \boldsymbol{v} - \nabla \times \nabla \phi = \boldsymbol{\omega} \,. \tag{3.39}$$

Furthermore, by using the vector identity

$$\nabla \cdot \nabla \boldsymbol{\psi} = \nabla \nabla \cdot \boldsymbol{\psi} - \nabla \times \nabla \times \boldsymbol{\psi}$$

we may write

$$\nabla \cdot \nabla \boldsymbol{\psi} = -\boldsymbol{\omega} \,. \tag{3.40}$$

To find ϕ we can take $\phi = 0$ at infinity. To obtain $\boldsymbol{\psi}$ we can use Green's function for the Laplace equation and arrive at

$$\boldsymbol{\psi}(\boldsymbol{x}) = \int\limits_{\Omega} \frac{\boldsymbol{\omega}(\boldsymbol{y})}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \, \mathrm{d}\boldsymbol{y} \,. \tag{3.41}$$

On the other hand, knowing the vorticity $\boldsymbol{\omega}$, we may compute the incompressible velocity by

$$\boldsymbol{v}_{\mathrm{ic}} = \nabla_x \times \int\limits_{\Omega} \frac{\boldsymbol{\omega}(\boldsymbol{y})}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \, \mathrm{d}\boldsymbol{y} = \int\limits_{\Omega} \frac{(\boldsymbol{y} - \boldsymbol{x}) \times \boldsymbol{\omega}(\boldsymbol{y})}{4\pi |\boldsymbol{x} - \boldsymbol{y}|^3} \, \mathrm{d}\boldsymbol{y} \,, \tag{3.42}$$

which is a pure kinematic relation. Now, because vorticity is transported by convection and diffusion, an initially confined region of vorticity will tend to remain within a bounded body, so that it may be assumed that $\boldsymbol{\omega} \to 0$ as $|\boldsymbol{x}| \to \infty$ with $O(1/|\boldsymbol{x}|^3)$ [5].

Furthermore, by assuming constant density ρ_0 the main part of Lighthills' source term for low Mach number flows may be rewritten by

$$\rho_0 \frac{\partial^2 v_{\mathrm{ic},i} v_{\mathrm{ic},j}}{\partial x_i \partial x_j} = \rho_0 \nabla \cdot (\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{ic}}) + \rho_0 \nabla \cdot \nabla \left(\frac{1}{2} \boldsymbol{v}_{\mathrm{ic}} \cdot \boldsymbol{v}_{\mathrm{ic}} \right) . \tag{3.43}$$

So, the overall fluctuating pressure can be seen as a superposition obtained by the two source terms

$$p'(\boldsymbol{x}, t) = p'_1 \left(\rho_0 \nabla \cdot (\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{ic}}) \right) + p'_2 \left(\rho_0 \nabla \cdot \nabla \left(\frac{1}{2} \boldsymbol{v}_{\mathrm{ic}} \cdot \boldsymbol{v}_{\mathrm{ic}} \right) \right) .$$

As shown in [5], one can derive the following estimates for the far field

$$p_{\mathrm{a1}} \sim \frac{l}{|\boldsymbol{x}|} \rho_0 u^2 \mathrm{Ma}^2 \tag{3.44}$$

$$p_{\mathrm{a2}} \sim \frac{l}{|\boldsymbol{x}|} \rho_0 u^2 \mathrm{Ma}^4 + \frac{l}{|\boldsymbol{x}|} \frac{\rho_0 u^2 \mathrm{Ma}^2}{Re} . \tag{3.45}$$

Therefore, we can state $p_{\mathrm{a2}} \ll p_{\mathrm{a1}}$ in turbulent flows, where $\mathrm{Ma} \ll 1$ and $Re \gg 1$. Furthermore, we can conclude that in such cases the component

$$\rho_0 \nabla \cdot (\boldsymbol{\omega} \times \boldsymbol{v}^{\mathrm{ic}}) \quad \text{of the Lighthill source term} \quad \rho_0 \frac{\partial^2 v_{\mathrm{ic},i} v_{\mathrm{ic},j}}{\partial x_i \partial x_j} \tag{3.46}$$

is the principle source of sound.

3.4. Perturbation Equations

The acoustic/viscous splitting technique for the prediction of flow induced sound was first introduced in [9], and afterwards many groups presented alternative and improved formulations for linear and non linear wave propagation [10, 11, 12, 13]. These formulations are all based on the idea, that the flow field quantities are split into compressible and incompressible parts.

For our derivation, we introduce a generic splitting of physical quantities to the conservation equations. For this purpose, we choose a combination of the two splitting approaches introduced above and define the following

$$p = \bar{p} + p_{\mathrm{ic}} + p_{\mathrm{c}} = \bar{p} + p_{\mathrm{ic}} + p_{\mathrm{a}} \tag{3.47}$$

$$\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{v}_{\mathrm{ic}} + \boldsymbol{v}_{\mathrm{c}} = \bar{\boldsymbol{v}} + \boldsymbol{v}_{\mathrm{ic}} + \boldsymbol{v}_{\mathrm{a}} \tag{3.48}$$

$$\rho = \rho_0 + \rho_1 + \rho_{\mathrm{a}} . \tag{3.49}$$

Thereby the field variables are split into mean and fluctuating parts just like in the LEE. In addition the fluctuating field variables are split into acoustic and non-acoustic components. Finally, the density correction ρ_1 is build in as introduced above. This choice is motivated by the following assumptions

- The acoustic field is a fluctuating field.
- The acoustic field is irrotational, i.e. $\nabla \times \boldsymbol{v}_\mathrm{a} = 0$.
- The acoustic field requires compressible media and an incompressible pressure fluctuation is not equivalent to an acoustic pressure fluctuation.

By doing so, we arrive for an incompressible flow at the following perturbation equations[1]

$$\frac{\partial p_\mathrm{a}}{\partial t} + \overline{\boldsymbol{v}} \cdot \nabla p_\mathrm{a} + \rho_0 c_0^2 \nabla \cdot \boldsymbol{v}_\mathrm{a} = -\frac{\partial p_\mathrm{ic}}{\partial t} - \overline{\boldsymbol{v}} \cdot \nabla p_\mathrm{ic} \tag{3.50}$$

$$\rho_0 \frac{\partial \boldsymbol{v}_\mathrm{a}}{\partial t} + \rho_0 \nabla\big(\overline{\boldsymbol{v}} \cdot \boldsymbol{v}_\mathrm{a}\big) + \nabla p_\mathrm{a} = 0 \tag{3.51}$$

with spatial constant mean density ρ_0 and speed of sound c_0. This system of partial differential equations is equivalent to the previously published ones [11]. The source term is the substantial derivative of the incompressible flow pressure p_ic. Using the acoustic scalar potential ψ_a and assuming a spacial constant mean density and speed of sound, we may rewrite (3.51) by

$$\nabla \left(\rho_0 \frac{\partial \psi_\mathrm{a}}{\partial t} + \rho_0 \, \overline{\boldsymbol{v}} \cdot \nabla \psi_\mathrm{a} - p_\mathrm{a} \right) = 0 \,, \tag{3.52}$$

and arrive at

$$p_\mathrm{a} = \rho_0 \frac{\partial \psi_\mathrm{a}}{\partial t} + \rho_0 \, \overline{\boldsymbol{v}} \cdot \nabla \psi_\mathrm{a} \,. \tag{3.53}$$

Now, we substitute (3.53) into (3.50) and arrive at

$$\frac{1}{c_0^2} \frac{D^2 \psi_\mathrm{a}}{Dt^2} - \Delta \psi_\mathrm{a} = -\frac{1}{\rho_0 c_0^2} \frac{D p_\mathrm{ic}}{Dt} \, ; \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{\boldsymbol{v}} \cdot \nabla \,. \tag{3.54}$$

This convective wave equation fully describes acoustic sources generated by incompressible flow structures and its wave propagation through flowing media. In addition, instead of the original unknowns p_a and $\boldsymbol{v}_\mathrm{a}$ we have know just the scalar unknown ψ_a. In accordance to the acoustic perturbation equations (APE), we name this resulting partial differential equation for the acoustic scalar potential as *Perturbed Convective Wave Equation* (PCWE).

Finally, it is of great interest that by neglecting the mean flow $\bar{\boldsymbol{v}}$ in (3.50), we arrive at the linearized conservation equations of acoustics with $\partial p_\mathrm{ic}/\partial t$ as a source term

$$\frac{1}{\rho_0 c_0^2} \frac{\partial p_\mathrm{a}}{\partial t} + \nabla \cdot \boldsymbol{v}_\mathrm{a} = \frac{-1}{\rho_0 c_0^2} \frac{\partial p_\mathrm{ic}}{\partial t} \tag{3.55}$$

$$\frac{\partial \boldsymbol{v}_\mathrm{a}}{\partial t} + \frac{1}{\rho_0} \nabla p_\mathrm{a} = 0 \,. \tag{3.56}$$

[1] For a detailed derivation of perturbation equations both for compressible as well as incompressible flows, we refer to [15]

As in the standard acoustic case, we apply $\partial/\partial t$ to (3.55) and $\nabla\cdot$ to (3.56) and subtract the two resulting equations to arrive at

$$\frac{1}{c_0^2}\frac{\partial^2 p_{\mathrm{a}}}{\partial t^2} - \nabla\cdot\nabla p_{\mathrm{a}} = \frac{-1}{c_0^2}\frac{\partial^2 p_{\mathrm{ic}}}{\partial t^2}\,. \tag{3.57}$$

We call this partial differential equation the aeroacoustic wave equation (AWE). Please note, that this equation can also be obtained by starting at Lighthill's inhomogeneous wave equation for incompressible flow, where we can substitute the second spatial derivative of Lighthill's tensor by the Laplacian of the incompressible flow pressure (see (3.13)). Using the decomposition of the fluctuating pressure p'

$$p' = p_{\mathrm{ic}} + p_{\mathrm{a}}\,.$$

results again into (3.57).

3.5. Comparison of Different Aeroacoustic Analogies

As a demonstrative example to compare the different acoustic analogies, we choose a cylinder in a cross flow, as displayed in Fig. 3.4. Thereby, the computational grid is just up to the

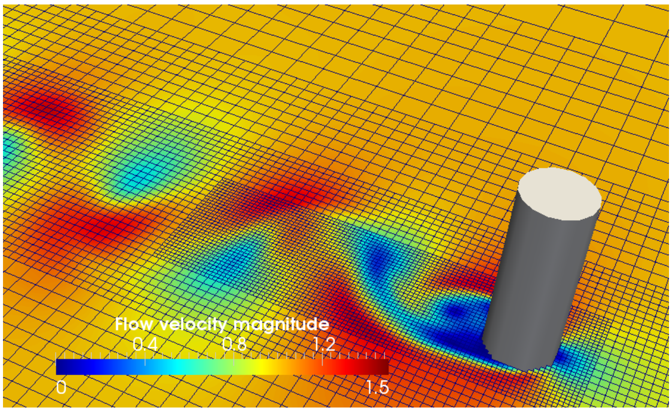


Figure 3.4.: Computational setup for flow computation.

height of the cylinder and together with the boundary conditions (bottom and top as well as span-wise direction symmetry boundary condition), we obtain a pseudo two-dimensional flow field. The diameter of the cylinder D is 1 m resulting with the inflow velocity of 1 m/s and chosen viscosity in a Reynolds number of 250 and Mach number of 0.2. From the flow simulations, we obtain a shedding frequency of 0.2 Hz (Strouhal number of 0.2). The acoustic mesh is chosen different from the flow mesh, and resolves the wavelength of two times the shedding frequency with 10 finite elements of second order. At the outer boundary of the acoustic domain we add a perfectly matched layer to efficiently absorb the outgoing waves. For the acoustic field computation we use the following formulations:

- Lighthill's acoustic analogy with Lighthill's tensor $[\boldsymbol{L}]$ according to (3.12) as source term

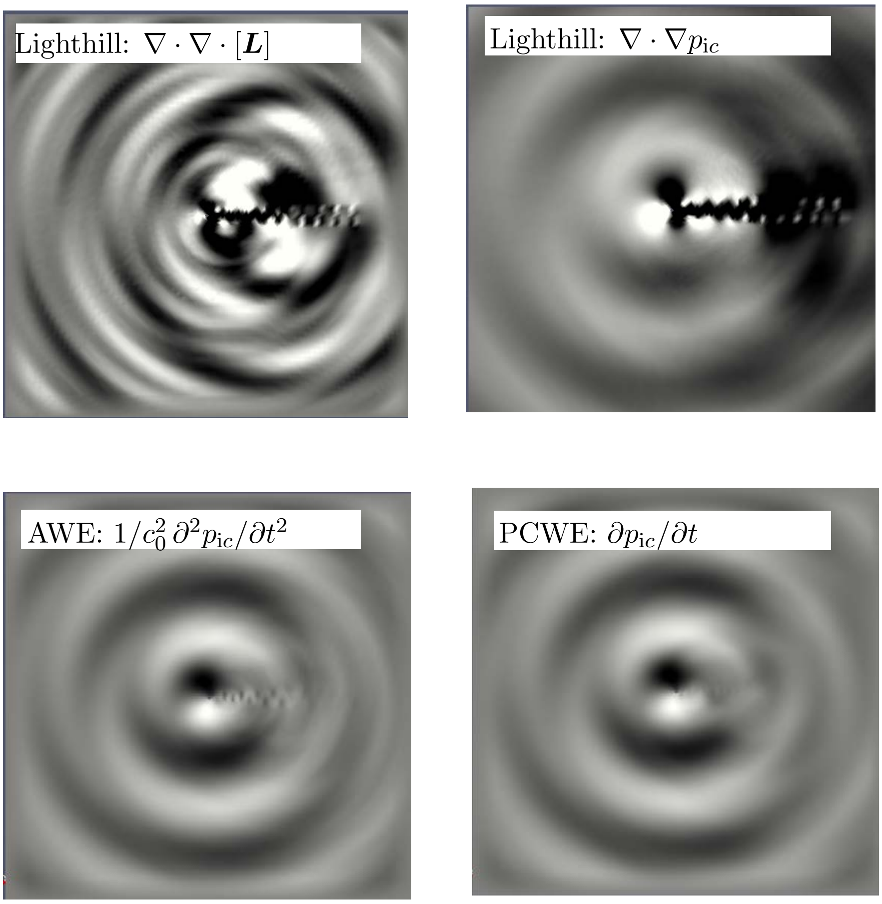


Figure 3.5.: Computed acoustic field with the different formulations.

- Lighthill's acoustic analogy with the Laplacian of the incompressible flow pressure p_{ic} as source term (see (3.13))

- the aeroacoustic wave equation (AWE) according to (3.57)

- Perturbed Convective Wave Equation (PCWE) according to (3.54); for comparison, we set the mean flow velocity $\bar{\boldsymbol{v}}$ to zero.

Figure 3.5 displays the acoustic field for the different formulations. One can clearly see that the acoustic field of PE (for comparison with the other formulations we have neglected the convective terms) meets very well the expected dipole structure and is free from dynamic flow disturbances. Furthermore, the acoustic field of AWE is quite similar and exhibits almost no dynamic flow disturbances. Both computations with Lighthill's analogy show flow disturbances, whereby the formulation with the Laplacian of the incompressible flow pressure as source term shows qualitative better result as the classical formulation based on the incompressible flow velocities.

3.6. Questions: Chapter 3

1. Explain the great idea of Lighthill towards his aeroacoustic analogy.
2. Derive Lighthill's inhomogeneous wave equation by using the following conservation equations

$$\begin{aligned} \frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot \boldsymbol{\pi} &= 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} &= 0 \end{aligned}$$

   and the momentum flux tensor $\pi_{ij} = \rho v_i v_j + (p - p_0)\delta_{ij} - \tau_{ij}$.
3. The acoustic pressure in the far field, generated by one vortex, can be estimated by

$$p^{\mathrm{a}} \sim \frac{l}{|\boldsymbol{x}|} \frac{\rho_0 v^4}{c_0^2}$$

   Compute the acoustic power and show the scaling towards the Mach number!
4. Describe the idea of Curle and his method to include in his integral formulation the generation of sound by flows around obstacles.
5. The acoustic pressure in the far field, generated by one vortex in the presence of an obstacle, can be estimated by

$$p^{\mathrm{a}} \sim \frac{1}{c_0|\boldsymbol{x}|} \rho_0 v^3 .$$

   Compute the acoustic power and show the scaling towards the Mach number!
6. Proof the following relation

$$\rho_0 \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} = \rho_0 \nabla \cdot (\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{ic}}) + \rho_0 \nabla \cdot \nabla \left(\frac{1}{2} (v_{\mathrm{ic}})^2 \right)$$

   Which term is the dominat one towards the acoustic pressure in the far field?
7. Show that for an incompressible fluid the following relation holds

$$\nabla \cdot \nabla p_{\mathrm{ic}} = -\rho_0 \frac{\partial^2 v_{\mathrm{ic},i} v_{\mathrm{ic},j}}{\partial x_i \partial x_j} .$$

   Here, p_{ic} and $\boldsymbol{v}_{\mathrm{ic}}$ denotes the incompressible flow pressure and velocity.
8. Derive the aeroacoustic wave equation (AWE) by starting at Lighthill's inhomogeneous wave equation (with the Laplacian of the incompressible flow pressure as source term)

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla \cdot \nabla p' = -\nabla \cdot \nabla p_{\mathrm{ic}}$$

   and substitute the decomposition of the fluctuating pressure

$$p' = p_{\mathrm{ic}} + p_{\mathrm{a}} .$$

   Does AWE consider convection and refraction effects?

4. One dimensional acoustics

4.1. Plane waves

For many practical applications, e.g., duct acoustics as occurring in air-conditioning systems (trains, automotive, rooms, etc.), musical acoustics, speech production, etc., we can assume a plane wave assumption. To be precise, we have a lower and upper frequency limit defined by

$$\frac{2\nu}{\pi d^2} \ll f < \frac{c_0}{2d} \,. \tag{4.1}$$

Thereby, $\nu = \mu/\rho$ is the kinematic viscosity and d the pipe width (or diameter). The upper limit is given by the critical (also called cut-off) frequency f_g in a hard-walled pipe. Above this frequency we will also have wave propagation in radial direction (see Fig. 4.1). The

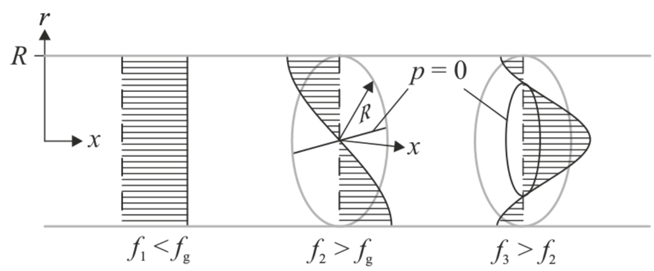


Figure 4.1.: Acoustic pressure distribution in hard-walled pipes [16].

lower limit is given by the condition of frictionless wave propagation. Thereby, the effect of viscosity is confined to boundary layers of thickness δ_A near walls, which computes by

$$\delta_\mathrm{A} = \sqrt{\frac{\mu}{\pi \rho f}} \,. \tag{4.2}$$

In order to make a plane wave approximation reasonable, we should have thin viscous boundary layers with $\delta_\mathrm{A}/d \ll 1$. Assuming a kinematic viscosity of $1.5 \cdot 10^{-5}\,\mathrm{m}^2/\mathrm{s}$ in air, the plane wave approximation is valid over three decades of audio range within a pipe of diameter $d = O(10^{-2})\mathrm{m}$.[1]

A main topic in duct acoustics is the modelling of sources. Here, we will concern sound generation in compact regions as a result of sudden changes in cross section or localized fluid injection. Thereby, in our case a region with a length of the order of the pipe width d will be defined compact. Such regions we can treat separately, e.g., by solving the full set of partial differential equations in case of an axial fan. If the end of the pipe is part of the problem, we will consider this end by an impedance.

[1] O denotes the order.

4.2. Standing waves and resonance

We consider a closed pipe, in which an acoustic pressure distribution (in the frequency) simply computes by the superposition of propagating and reflected waves

$$\hat{p}_\mathrm{a} = p^+ \left(e^{-jkx} + Re^{jkx} \right) . \tag{4.3}$$

Thereby, R denotes the reflection coefficient (see (2.45)). The particle velocity computes according to the linear momentum conservation by

$$\hat{v}_\mathrm{a}(x) = \frac{p^+}{\rho_0 c_0} \left(e^{-jkx} - Re^{jkx} \right) \tag{4.4}$$

Let's assume, that the pipe is terminated hard-walled, which means $\hat{v}_\mathrm{a} = 0$, and let's put at this position the origin $x = 0$. In this case the reflection coefficient R is one, and we obtain for the acoustic pressure

$$\hat{p}_\mathrm{a}(x) = 2p^+ \cos(kx)$$

and for the acoustic particle velocity

$$\hat{v}_\mathrm{a}(x) = \frac{-j2p^+}{\rho_0 c_0} \sin(kx) .$$

Now, the space-time distribution is obtained by taking the real part of the complex valued quantities

$$\begin{aligned} p_\mathrm{a}(x,t) &= \mathrm{Re}\left(\hat{p}_\mathrm{a} e^{j\omega t}\right) = 2p^+ \cos(kx) \cos(\omega t) & (4.5) \\ v_\mathrm{a}(x,t) &= \mathrm{Re}\left(\hat{v}_\mathrm{a} e^{j\omega t}\right) = \frac{2p^+}{\rho_0 c_0} \sin(kx) \sin(\omega t) . & (4.6) \end{aligned}$$

Such a wave is called a standing wave, since the spatial function is constant and the local amplitude is modulated by the time dependent term (see Fig.4.2). Let's assume that the

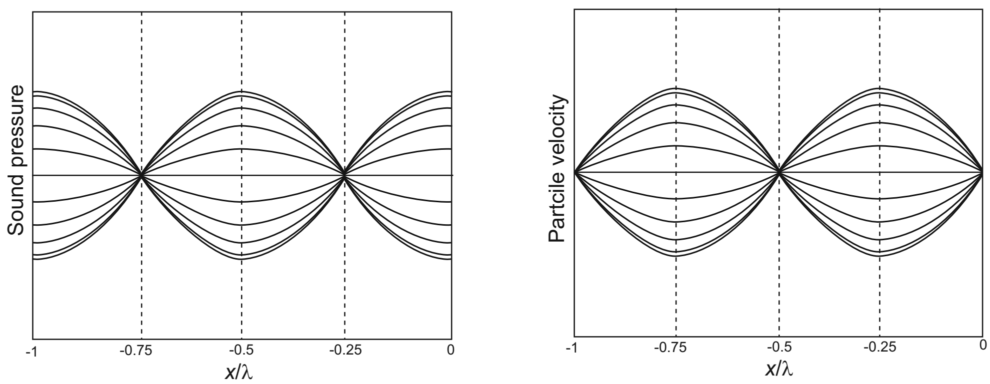


Figure 4.2.: Spatial distribution of acoustic pressure and particle velocity for constant times.

standing wave is excited by a piston at $x = -l$ moving with velocity $\hat{v}_\mathrm{p}$. Then, we may write

$$\hat{v}_\mathrm{p} = \hat{v}_\mathrm{a}(x = -l) = \frac{j2p_\mathrm{a}^+}{\rho_0 c_0} \sin(kl)$$

and obtain for the pressure amplitude

$$p_\mathrm{a}^+ = \frac{-j\rho_0 c_0 \hat{v}_\mathrm{a}}{2\sin(kl)}\,.$$

Note that for $kl = n\pi$ with $n \in \{1,2,3..\}$ the denominator gets zero and the amplitude of the acoustic pressure to infinity. Using $k = \omega/c_0 = 2\pi/\lambda$, we obtain the resonance frequencies

$$f = \frac{nc_0}{2l}\,; \quad l = n\,\frac{\lambda}{2}\,. \tag{4.7}$$

Therefore, for pipes with a length l of integer multiples of half the wavelength λ, we obtain resonance.

4.3. Two-terminal-pair networks

According to (2.43) and (2.44), the acoustic pressure and particle velocity in a tube computes by

$$\hat{p}_\mathrm{a}(x) = p^+ e^{-jkx} + p^- e^{jkx} \tag{4.8}$$

$$\hat{v}_\mathrm{a}(x) = \frac{1}{\rho_0 c_0}\left(p^+ e^{-jkx} - p^- e^{jkx}\right)\,. \tag{4.9}$$

At $x = 0$, we obtain

$$\hat{p}_\mathrm{a}(0) = \check{p} = p^+ + p^-\,; \quad \hat{v}_\mathrm{a}(0) = \check{v} = \frac{p^+ - p^-}{\rho_0 c_0}\,.$$

Therefore, we may write the transfer equations with $Z_0 = \rho_0 c_0$ by

$$\hat{p}_\mathrm{a}(x) = \check{p}\cos(kx) - jZ_0\check{v}\sin(kx) \tag{4.10}$$

$$\hat{v}_\mathrm{a}(x) = -j\frac{\check{p}}{Z_0}\sin(kx) + \check{v}\cos(kx)\,. \tag{4.11}$$

For a tube with length l and $p_{\mathrm{a},1}$, $v_{\mathrm{a},1}$ at the left side and $p_{\mathrm{a},2}$, $v_{\mathrm{a},2}$ on the right side as

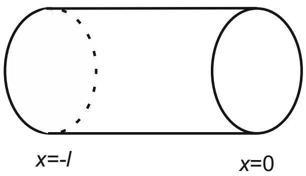


Figure 4.3.: Simple duct.

displayed in Fig. 4.3, we obtain from (4.10), (4.11) with $x = -l$

$$\begin{pmatrix} \hat{p}_{\mathrm{a},1} \\ \hat{v}_{\mathrm{a},1} \end{pmatrix} = \begin{pmatrix} \cos(kl) & jZ_0\sin(kl) \\ \frac{j\sin(kl)}{Z_0} & \cos(kl) \end{pmatrix} \begin{pmatrix} \hat{p}_{\mathrm{a},2} \\ \hat{v}_{\mathrm{a},2} \end{pmatrix}\,. \tag{4.12}$$

Now, in a lumped element model for any physical field one chooses a potential quantity and a flux quantity and its multiplication results in the power of the system. Therefore, we use instead the acoustic particle velocity the acoustic volume flux

$$q_\mathrm{a} = \int_A \boldsymbol{v}_\mathrm{a} \cdot \mathrm{d}\boldsymbol{s} = v_\mathrm{a} A \tag{4.13}$$

with A the cross section and indeed for duct acoustics we obtain

$$P_\mathrm{a} = \int\limits_A p_\mathrm{a}\boldsymbol{v}_\mathrm{a} \cdot \mathrm{d}\boldsymbol{s} = p_\mathrm{a} q_\mathrm{a} \,. \tag{4.14}$$

Finally, (4.12) changes to

$$\begin{pmatrix} \hat{p}_{\mathrm{a},1} \\ \hat{q}_{\mathrm{a},1} \end{pmatrix} = \begin{pmatrix} \cos(kl) & j\frac{Z_0}{A}\sin(kl) \\ j\frac{A}{Z_0}\sin(kl) & \cos(kl) \end{pmatrix} \begin{pmatrix} \hat{p}_{\mathrm{a},2} \\ \hat{q}_{\mathrm{a},2} \end{pmatrix} = \boldsymbol{K} \begin{pmatrix} \hat{p}_{\mathrm{a},2} \\ \hat{q}_{\mathrm{a},2} \end{pmatrix} . \tag{4.15}$$

Now, let us apply this formalism to a jump in cross section as displayed in Fig. 4.4. To get

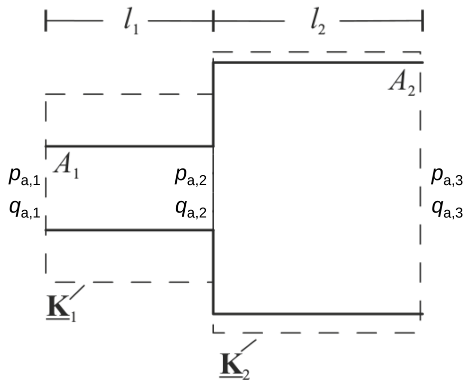


Figure 4.4.: Duct with a jump in cross section.

the transfer function between input and output, we just have to compute the transfer matrix $\boldsymbol{K}_i$ for each individual duct and the overall transfer matrix $\boldsymbol{K}_\mathrm{total}$ by its multiplication.

$$\begin{aligned} \begin{pmatrix} \hat{p}_{\mathrm{a},1} \\ \hat{q}_{\mathrm{a},1} \end{pmatrix} &= \boldsymbol{K}_1\ \boldsymbol{K}_2 \begin{pmatrix} \hat{p}_{\mathrm{a},3} \\ \hat{q}_{\mathrm{a},3} \end{pmatrix} \\ &= \begin{pmatrix} \cos(kl_1) & j\frac{Z_0}{A}\sin(kl_1) \\ j\frac{A_1}{Z_0}\sin(kl_1) & \cos(kl_1) \end{pmatrix} \begin{pmatrix} \cos(kl_2) & j\frac{Z_0}{A_2}\sin(kl_2) \\ j\frac{A_2}{Z_0}\sin(kl_2) & \cos(kl_2) \end{pmatrix} \begin{pmatrix} \hat{p}_{\mathrm{a},3} \\ \hat{q}_{\mathrm{a},3} \end{pmatrix} . \end{aligned} \tag{4.16}$$

Figure 4.5 demonstrates the chain of the two-terminal-pair network. In general, we can

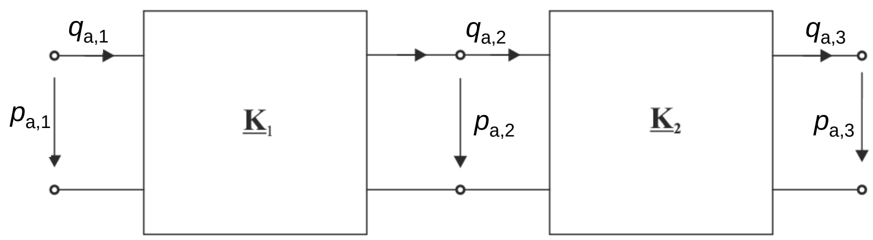


Figure 4.5.: Two-terminal-pair network for cross section jump.

compute the transfer function between input and output for any number N of sub-systems as follows

$$\begin{pmatrix} \hat{p}_{\mathrm{a,in}} \\ \hat{q}_{\mathrm{a,in}} \end{pmatrix} = \prod_{i=1}^{N} \boldsymbol{K}_i \begin{pmatrix} \hat{p}_{\mathrm{a},N} \\ \hat{q}_{\mathrm{a},N} \end{pmatrix} \tag{4.17}$$

In a next step, we consider the case of a side-branch as displayed in Fig. 4.6. The sound

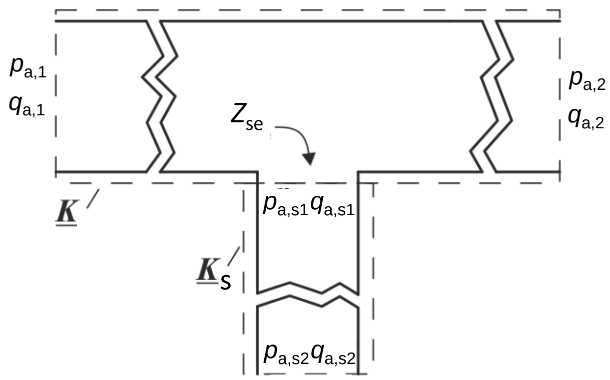


Figure 4.6.: Two-terminal-pair network with one side branch.

field in the main branch is effected by the side-branch and therefore, we have to compute in a first step the transfer matrix $\boldsymbol{K}_{\mathrm{s}}$ of the side-branch. To this end, we have the following relation

$$\begin{pmatrix} \hat{p}_{\mathrm{a,s1}} \\ \hat{q}_{\mathrm{a,s1}} \end{pmatrix} = \boldsymbol{K}_{\mathrm{s}} \begin{pmatrix} \hat{p}_{\mathrm{a,s2}} \\ \hat{q}_{\mathrm{a,s2}} \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \hat{p}_{\mathrm{a,s2}} \\ \hat{q}_{\mathrm{a,s2}} \end{pmatrix} . \tag{4.18}$$

Now, there are different possibilities, how the side-branch is terminated. Let us discuss two practical relevant cases. In the first case, the side-branch may be terminated hard-walled. Then, the acoustic volume flux $q_{\mathrm{a,s2}}$ is zero (open circuit), and the input impedance $\hat{Z}_{\mathrm{se}}$ computes by

$$\hat{Z}_{\mathrm{se}} = \frac{\hat{p}_{\mathrm{a,s1}}}{\hat{q}_{\mathrm{a,s1}}} = \frac{K_{11}}{K_{21}} . \tag{4.19}$$

When the acoustic pressure $p_{\mathrm{a,s2}}$ is zero (short-cut), then the acoustic impedance computes by

$$\hat{Z}_{\mathrm{se}} = \frac{\hat{p}_{\mathrm{a,s1}}}{\hat{q}_{\mathrm{a,s1}}} = \frac{K_{12}}{K_{22}} . \tag{4.20}$$

In both cases the acoustic impedance $\hat{Z}_{\mathrm{se}}$ is in parallel and the transfer matrix K_{s} computes by

$$\boldsymbol{K}_{\mathrm{s}} = \begin{pmatrix} 1 & 0 \\ 1/\hat{Z}_{\mathrm{se}} & 1 \end{pmatrix} \tag{4.21}$$

For our investigation, we assume that the side-branch is hard-walled terminated (see Fig. 4.7). According to (4.12) and setting $\hat{v}_{\mathrm{a},2}$ zero, we obtain the following relation

$$\begin{pmatrix} \hat{p}_{\mathrm{a,s1}} \\ \hat{q}_{\mathrm{a,s1}} \end{pmatrix} = \begin{pmatrix} \cos(kl_{\mathrm{s}}) & j\frac{Z_0}{A_{\mathrm{s}}}\sin(kl_{\mathrm{s}}) \\ j\frac{A_{\mathrm{s}}}{Z_0}\sin(kl_{\mathrm{s}}) & \cos(kl_{\mathrm{s}}) \end{pmatrix} \begin{pmatrix} p_{\mathrm{a,s2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(kl_{\mathrm{s}}) \\ j\frac{A_{\mathrm{s}}}{Z_0}\sin(kl_{\mathrm{s}}) \end{pmatrix} p_{\mathrm{a,s2}} \tag{4.22}$$

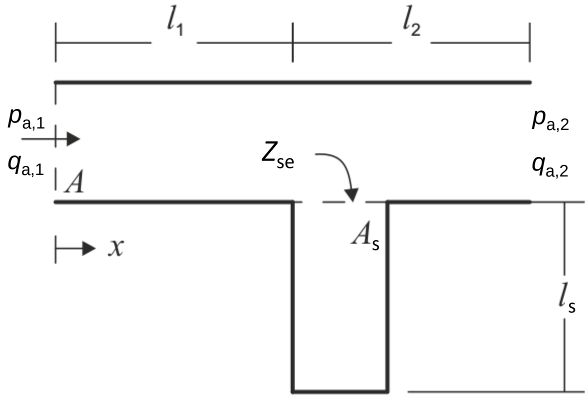


Figure 4.7.: Two-terminal-pair network with a side branch hard-walled terminated.

from which the input impedance computes to

$$\hat{Z}_{\mathrm{se}} = \frac{p_{\mathrm{a,s1}}}{q_{\mathrm{a,s1}}} = \frac{Z_0 \cot(kl_{\mathrm{s}})}{jA_{\mathrm{s}}}\,. \tag{4.23}$$

Now, the overall transfer chain is displayed in Fig. 4.8, and we obtain the following relation between input and output

$$\begin{aligned} \begin{pmatrix} p_{\mathrm{a},1} \\ q_{\mathrm{a},1} \end{pmatrix} &= \boldsymbol{K}_1\, \boldsymbol{K}(\hat{Z}_{\mathrm{e}})\, \boldsymbol{K}_2 \begin{pmatrix} p_{\mathrm{a},2} \\ q_{\mathrm{a},2} \end{pmatrix} \\ &= \begin{pmatrix} \cos(kl_1) & j\frac{Z_0}{A_1}\sin(kl_1) \\ j\frac{A_1}{Z_0}\sin(kl_1) & \cos(kl_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/\hat{Z}_{\mathrm{se}} & 1 \end{pmatrix} \\ & \quad \begin{pmatrix} \cos(kl_2) & j\frac{Z_0}{A_2}\sin(kl_2) \\ j\frac{A_2}{Z_0}\sin(kl_2) & \cos(kl_2) \end{pmatrix} \begin{pmatrix} p_{\mathrm{a},2} \\ q_{\mathrm{a},2} \end{pmatrix} \end{aligned} \tag{4.24}$$

Finally, we provide the radiation impedance for an open pipe of radius R [17]

$$\hat{Z}_{\mathrm{r}} = \frac{Z_0}{\pi R^2}\left(\frac{1}{4}(kR)^2 + j\frac{8kR}{3\pi\sqrt{2}}\right)\,. \tag{4.25}$$

In this case, the acoustic pressure at the end is set to zero and the impedance Z_{r} is in series to the previous network.

4.4. Source terms

In a compact region of length L and fixed volume Ω enclosed by the surface Γ, we apply the conservation laws of mass and momentum in integral form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \rho\;\mathrm{d}\boldsymbol{x} + \int_{\Gamma} \rho\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s = 0 \tag{4.26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \rho\boldsymbol{v}\;\mathrm{d}\boldsymbol{x} + \int_{\Gamma} (\boldsymbol{\sigma}_{\mathrm{f}} + \rho\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{\Omega} \boldsymbol{f}\;\mathrm{d}\boldsymbol{x}\,. \tag{4.27}$$

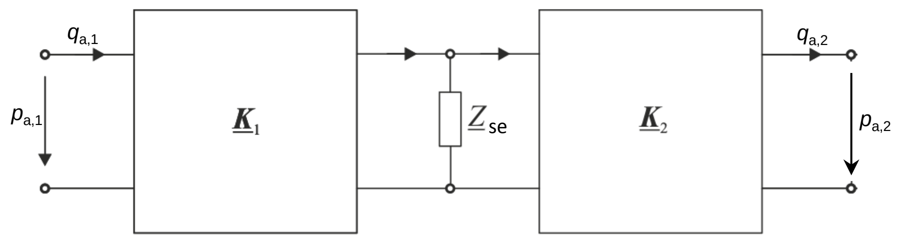


Figure 4.8.: Transfer chain for a two-terminal-pair network with a side branch hard-walled terminated.

In the considered volume Ω we describe the flow in three dimensional detail. Thereby, the force density $\boldsymbol{f}$ may model the effect of a complex flow such as the flow around a ventilator fan by assuming a localized momentum source. Neglecting the volume contribution of $\mathrm{d}/\,\mathrm{d}t \int_\Omega \rho \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$ as well as the nonlinearity (convective term) and viscosity, we obtain in a linear approximation in one dimension

$$f_x = \Delta p' \delta(x-y)\,. \tag{4.28}$$

Thereby, $\Delta p'$ is the in-stationary pressure difference between input and output surface (may be computed by CFD) and the function $\delta(x-y)$ models the compact region as a region of zero extension. This term can be using in a one dimensional wave equation (network representation) as a representation of a complex flow such as that around a ventilator.

4.5. Helmholtz resonator

The resonance condition for duct segments imply that the length of the tube is a multiple integer of half the wavelength (see (4.7)). Therefore, in many practical applications this would imply that resonators used to absorb sound should be quite large. However, a solution to this problem is to use a non-uniform pipe in the shape of a bottle as displayed in Fig. 4.9. If the cross-section area A_b of the bottle is large compared to A_n of the neck, then we can neglect the acoustic particle velocity and assume the density as constant in the bottle. Therefore, applying the linearized mass conservation for acoustics in integral form results in

$$\Omega \frac{\mathrm{d}\rho_\mathrm{a}}{\mathrm{d}t} + \rho_0 A_\mathrm{n} v_\mathrm{a,n} = 0\,. \tag{4.29}$$

Here, $v_\mathrm{a,n}$ denotes the acoustic particle velocity in the neck, which can be assumed to be constant within this region. Furthermore, using the linearized momentum conservation for

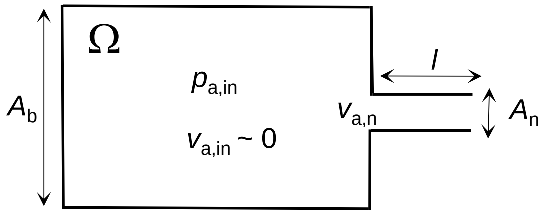


Figure 4.9.: Design of a Helmholtz resonator.

acoustics (integral form) in the neck provides

$$\rho_0 l \frac{\mathrm{d}v_{\mathrm{a,n}}}{\mathrm{d}t} + p_{\mathrm{a}} = 0\,. \tag{4.30}$$

However, just using l as the geometric length of the neck is not correct. We have to take into account the inertia of the acoustic particle velocity at both ends of the neck, so that a correction has to be performed. A standard correction is to add the term $R_{\mathrm{n}}\pi/2$ with R_{n} the radius of the neck, which changes (4.30) to

$$\rho_0 \left(l + \frac{R_{\mathrm{n}}\pi}{2} \right) \frac{\mathrm{d}v_{\mathrm{a,n}}}{\mathrm{d}t} + p_{\mathrm{a}} = 0\,. \tag{4.31}$$

Combining (4.29) and (4.31) with the standard pressure-density relation over the speed of sound c_0 results in a pure differential equation for the acoustic pressure

$$\omega \frac{(l + R_{\mathrm{n}}\pi/2)}{c_0^2 A_{\mathrm{n}}} \frac{\mathrm{d}^2 p_{\mathrm{a}}}{\mathrm{d}t^2} - p_{\mathrm{a}} = 0\,. \tag{4.32}$$

Thus, we obtain the resonance frequency by

$$f_{\mathrm{res}} = \frac{1}{2\pi} \sqrt{\frac{c_0^2 A_{\mathrm{n}}}{(l + R_{\mathrm{n}}\pi/2)\,\Omega}}\,. \tag{4.33}$$

4.6. Flow-induced oscillations by a Helmholtz resonator

In the previous sections, we have just considered the acoustic field in a quiescent fluid. However, very often the flow can not be neglected and furthermore, it is the cause for exciting acoustic fields. A quite famous example is flow induced pulsations of a Helmholtz resonator or wall cavity, e.g., sun-roof of a car.

The configuration which we consider is displayed in Fig. 4.10. Self-sustained oscillations with a frequency close to the resonance frequency of the resonator occur when the phase condition for a perturbation in the feedback loop (shear layer / resonator) is satisfied and the gain is large enough. In a first order approximation, perturbations of the shear layer at the opening of the resonator propagate with a velocity v_{c} close to $0.4v_0$. Thereby, it appears from experiments that when the travelling time of a perturbation across the opening with

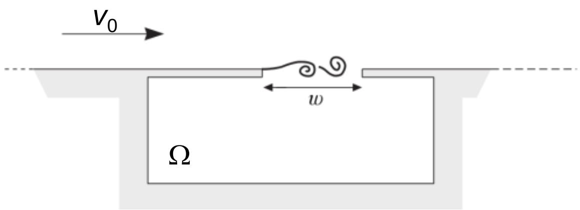


Figure 4.10.: Helmholtz resonator in a wall with grazing flow [18].

length w roughly matches the oscillation period $2\pi/\omega_{\mathrm{res}}$ of the resonator (or integer multiples n) pulsation occurs

$$T_{\mathrm{travel}} = \frac{w}{0.4v_0} = \frac{2\pi}{\omega_{\mathrm{res}}} n \;\rightarrow\; 2\pi n = \frac{w\,\omega_{\mathrm{res}}}{0.4v_0}\,. \tag{4.34}$$

The vorticity of the shear layer is concentrated into discrete vortices. At a moderate amplitude of the acoustic particle velocity $v_{\mathrm{a}}/v_0 = O(0.1)$ one can assume that the acoustic field only triggers the flow instability but does not dramatically modify the amount of vorticity shed at the upstream edge of the slot. Using Howe's analogy, one can compute the acoustic power P_{a} generated by vortices due to instability of the grazing flow along the orifice of area $A = w\,b$ by [5]

$$P_{\mathrm{a}} = -\rho_0 \int_{\Omega} \overline{(\boldsymbol{\omega} \times \boldsymbol{v}) \cdot \boldsymbol{v}_{\mathrm{a}}}\, \mathrm{d}\boldsymbol{x} \sim O(5 \cdot 10^{-2})\, \frac{1}{2}\rho_0 v_0^2 w b v_{\mathrm{a}}\,. \tag{4.35}$$

Upon formation of a new vortex, the acoustic particle velocity $\boldsymbol{v}_{\mathrm{a}}$ is directed towards the

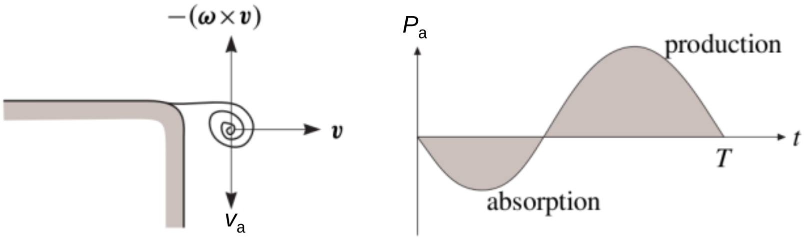


Figure 4.11.: Absorption and production of acoustic energy by vortex shedding [18].

interior of the resonator (see Fig. 4.11). Therefore, initially the vortex is absorbing energy from the acoustic field. In the second half of the acoustic period the acoustic particle velocity $\boldsymbol{v}_{\mathrm{a}}$ is directed outwards from the resonator, and acoustic energy is produced.

4.7. Questions: Chapter 4

1. For duct acoustic to be considered one dimensional, we have a lower and upper frequency limit. Describe the physical effects of these limits.

2. Assuming one dimensional acoustics in a duct, derive the formula for the acoustic pressure and the acoustic particle velocity.

3. A pipe segment with a different cross sectional area A_2 than the cross section A_1 of the rest of the pipe can be used as a filter to prevent the propagation of waves generated by a piston. Two solutions can be considered: $A_2 > A_1$ and $A_1 < A_2$ (see Fig. 4.12). Assuming an ideal open end at $x = L1 + L2 + L3$, provide a set of equations from which we can compute the acoustic particle velocity v_a at the end of the pipe for a given piston velocity v_p.

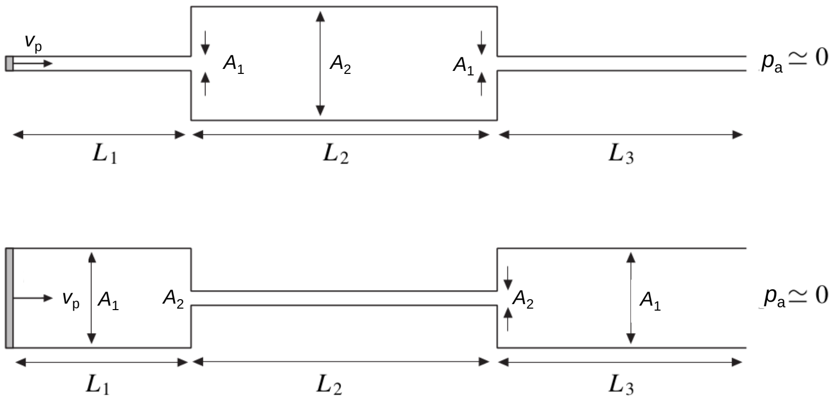


Figure 4.12.: Pipe segment

4. Proof that for an acoustic impedance $\hat{Z}_{se}$ being parallel, the transfer matrix $\boldsymbol{K}_s$ computes by (4.21).

Appendices

A. Vector identities and operations in different coordinate systems

The nabla operator is defined by

$$\nabla = \frac{\partial}{\partial x}\,\boldsymbol{e}_x + \frac{\partial}{\partial y}\,\boldsymbol{e}_y + \frac{\partial}{\partial z}\,\boldsymbol{e}_z \tag{A.1}$$

with the unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$. Thereby, the gradient of a scalar function ϕ results in a vector field and computes by

$$\nabla\phi = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{pmatrix} . \tag{A.2}$$

The divergence of a vector field results in a scalar value

$$\nabla\cdot\boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} . \tag{A.3}$$

Finally, the curl of a vector $\boldsymbol{u}$ computes as

$$\nabla\times\boldsymbol{u} = \begin{pmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{pmatrix} = \begin{pmatrix} \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \\ \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \end{pmatrix} . \tag{A.4}$$

In addition, the gradient of a vector $\boldsymbol{u}$ computes by

$$\nabla\boldsymbol{u} = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} . \tag{A.5}$$

By using the definitions of gradient, divergence, and curl, the following relations hold

$$\nabla(\xi\eta) = \xi\nabla\eta + \eta\nabla\xi \tag{A.6}$$

$$\nabla\cdot(\xi\boldsymbol{u}) = \xi\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\xi \tag{A.7}$$

$$\nabla\cdot(\boldsymbol{u}_1\times\boldsymbol{u}_2) = \boldsymbol{u}_2\cdot\nabla\times\boldsymbol{u}_1 - \boldsymbol{u}_1\cdot\nabla\times\boldsymbol{u}_2 \tag{A.8}$$

$$\nabla\times(\xi\boldsymbol{u}) = \xi\nabla\times\boldsymbol{u} - \boldsymbol{u}\times\nabla\xi \tag{A.9}$$

$$\nabla\cdot\nabla\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u}) \tag{A.10}$$

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{v}) = (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{u} - \boldsymbol{v}\,(\nabla\cdot\boldsymbol{u}) - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v} + \boldsymbol{u}\,(\nabla\cdot\boldsymbol{v}) \tag{A.11}$$

$$\xi\,\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\cdot(\xi\boldsymbol{u}\boldsymbol{u}) - \boldsymbol{u}\nabla\cdot\xi\boldsymbol{u} \tag{A.12}$$

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\times\boldsymbol{u}\,\times\boldsymbol{u} + \nabla\frac{1}{2}u^2 . \tag{A.13}$$

These relations combine the essential differential operators and build up a basis for the description of physical fields.

Furthermore, we introduce the Laplace operator on a scalar

$$\nabla \cdot \nabla \phi = \triangle \phi = (\nabla \cdot \nabla)\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\,, \tag{A.14}$$

and for a vector $\boldsymbol{u}$

$$(\nabla \cdot \nabla)\boldsymbol{u} = \triangle \boldsymbol{u} = \begin{pmatrix} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \\ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \\ \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \end{pmatrix}. \tag{A.15}$$

The following operations always result in zero

$$\nabla \times (\nabla \phi) = 0\,; \quad \nabla \cdot (\nabla \times \nabla \boldsymbol{u}) = 0\,. \tag{A.16}$$

In a next step, we will define the above introduced vector operations in the cylindrical coordinate system. In doing so, we have (see Fig. A.1a)

$$x = r\cos\varphi\,; \quad y = r\sin\varphi\,; \quad z = z\,, \tag{A.17}$$

and therefore

$$\boldsymbol{u} = u_r\boldsymbol{e}_r + u_\varphi\boldsymbol{e}_\varphi + u_z\boldsymbol{e}_z \tag{A.18}$$

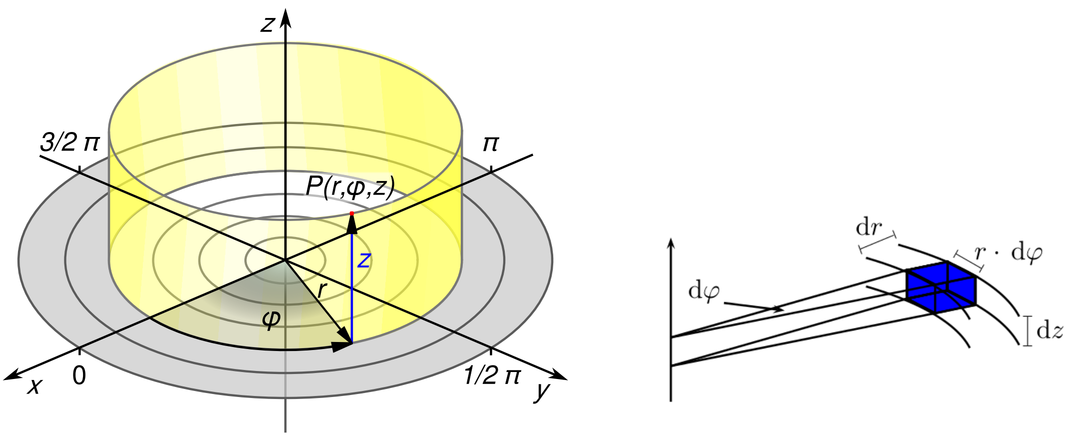
$$\boldsymbol{e}_r = \cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y + \boldsymbol{e}_z \tag{A.19}$$

$$\boldsymbol{e}_\varphi = -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y \tag{A.20}$$

$$\boldsymbol{e}_z = \boldsymbol{e}_z \tag{A.21}$$

$$\nabla = \boldsymbol{e}_r\frac{\partial}{\partial r} + \frac{\boldsymbol{e}_\varphi}{r}\frac{\partial}{\partial \varphi} + \boldsymbol{e}_z\frac{\partial}{\partial z}\,. \tag{A.22}$$

Therefore, we obtain for the gradient, divergence and curl operations in cylindrical coordi-



(a) Cylindrical coordinate system.

(b) Volume element.

Figure A.1.: Cylindrical coordinate systems.

nates the following formula

$$\nabla\phi = \begin{pmatrix} \frac{\partial\phi}{\partial r} \\ \frac{1}{r}\frac{\partial\phi}{\partial\varphi} \\ \frac{\partial\phi}{\partial z} \end{pmatrix} \tag{A.23}$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{\partial u_z}{\partial z} \tag{A.24}$$

$$\nabla\times\boldsymbol{u} = \frac{1}{r}\begin{pmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\varphi & \boldsymbol{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ u_r & ru_\varphi & u_z \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\left(\frac{\partial u_z}{\partial\varphi} - \frac{\partial(ru_\varphi)}{\partial r}\right) \\ r\frac{\partial u_r}{\partial z} - r\frac{\partial u_z}{\partial r} \\ \frac{1}{r}\left(\frac{\partial(ru_\varphi)}{\partial r} - \frac{\partial u_r}{\partial\varphi}\right) \end{pmatrix}. \tag{A.25}$$

Furthermore, the Laplacian of a scalar function ϕ computes by

$$\nabla\cdot\nabla\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{A.26}$$

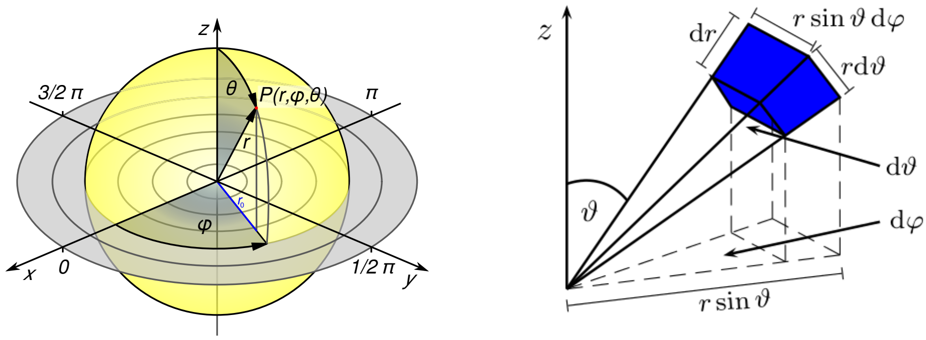
Performing an integration in cylindrical coordinates, needs a transformation for the volume element (see Fig. A.1b)

$$\mathrm{d}\Omega = r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z\,. \tag{A.27}$$

Therefore, we obtain

$$\int\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\limits_{z}\int\limits_{r}\int\limits_{\varphi} f(r,\varphi,z)\,r\,\mathrm{d}\varphi\,\mathrm{d}r\,\mathrm{d}z\,. \tag{A.28}$$

Furthermore, we also provide all these relations for spherical coordinates. Thereby, we have



(a) Spherical coordinate system.

(b) Volume element.

Figure A.2.: Spherical coordinate systems

the relations (see Fig. A.2a)

$$x = r\cos\varphi\sin\vartheta\,;\quad y = r\sin\varphi\sin\vartheta\,;\quad z = r\cos\vartheta\,, \tag{A.29}$$

and therefore

$$
\begin{aligned}
\boldsymbol{u} &= u_r \boldsymbol{e}_r + u_\varphi \boldsymbol{e}_\varphi + u_\vartheta \boldsymbol{e}_\vartheta && \text{(A.30)}\\
\boldsymbol{e}_r &= \sin\vartheta\,(\cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y) + \cos\vartheta \boldsymbol{e}_z && \text{(A.31)}\\
\boldsymbol{e}_\varphi &= -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y && \text{(A.32)}\\
\boldsymbol{e}_\vartheta &= \cos\vartheta\,(\cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y) - \sin\vartheta \boldsymbol{e}_z && \text{(A.33)}\\
\boldsymbol{e}_z &= \boldsymbol{e}_z && \text{(A.34)}\\
\nabla &= \boldsymbol{e}_r \frac{\partial}{\partial r} + \frac{\boldsymbol{e}_\vartheta}{r}\frac{\partial}{\partial \vartheta} + \frac{\boldsymbol{e}_\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\,. && \text{(A.35)}
\end{aligned}
$$

Therefore, we obtain for the gradient, divergence and curl operations in spherical coordinates the following formula

$$
\nabla\phi = \begin{pmatrix} \frac{\partial\phi}{\partial r} \\ \frac{1}{r}\frac{\partial\phi}{\partial\vartheta} \\ \frac{1}{r\sin\vartheta}\frac{\partial\phi}{\partial\varphi} \end{pmatrix} \qquad \text{(A.36)}
$$

$$
\nabla\cdot\boldsymbol{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial \sin\vartheta u_\vartheta}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial u_\varphi}{\partial\varphi} \qquad \text{(A.37)}
$$

$$
\nabla\times\boldsymbol{u} = \frac{1}{r^2\sin\vartheta}\begin{pmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\vartheta & r\sin\vartheta\,\boldsymbol{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\vartheta} & \frac{\partial}{\partial\varphi} \\ u_r & r u_\vartheta & r\sin\vartheta u_\varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{r^2\sin\vartheta}\left(\frac{\partial(r\sin\vartheta\,u_\varphi)}{\partial\vartheta} - \frac{\partial(r u_\vartheta)}{\partial\varphi}\right) \\ r\left(\frac{\partial u_r}{\partial\varphi} - \frac{\partial(r\sin\vartheta\,u_\varphi)}{\partial r}\right) \\ r\sin\vartheta\left(\frac{\partial(r u_\vartheta)}{\partial r} - \frac{\partial u_r}{\partial\vartheta}\right) \end{pmatrix} \text{(A.38)}
$$

Furthermore, the Laplacian of a scalar function ϕ computes by

$$
\nabla\cdot\nabla\phi = \triangle\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\phi}{\partial\varphi^2}\,. \qquad \text{(A.39)}
$$

Performing an integration in the spherical coordinate system, the volume element transforms (see Fig. A.2b)

$$
\mathrm{d}\Omega = (r\,\mathrm{d}\vartheta)\,(\,\mathrm{d}r)\,(r\sin\vartheta\,\mathrm{d}\varphi) = r^2\sin\vartheta\,\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\vartheta\,, \qquad \text{(A.40)}
$$

and therefore the integral

$$
\int_\Omega f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_r\int_\varphi\int_\vartheta f(r,\varphi,\vartheta)\,r^2\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\,\mathrm{d}r\,. \qquad \text{(A.41)}
$$

Important is also the Helmholtz decomposition, which states that each vector field $\boldsymbol{v}$ (e.g., the flow velocity) can be decomposed in an irrotational field described by the gradient of a scalar potential ϕ and in a solenoidal field $\boldsymbol{u}$ described by a vector potential $\boldsymbol{A}$

$$
\boldsymbol{v} = \nabla\times\boldsymbol{A} + \nabla\phi\,. \qquad \text{(A.42)}
$$

Furthermore, the integral theorem of Gauss (also known as the divergence theorem) transforms a volume integral to a surface integral

$$
\int_\Omega \nabla\cdot u\,\mathrm{d}\boldsymbol{x} = \oint_{\Gamma(\Omega)} \boldsymbol{u}\cdot\,\mathrm{d}\boldsymbol{s} = \oint_{\Gamma(\Omega)} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s\,. \qquad \text{(A.43)}
$$

The theorem of Stokes transforms a surface integral to a line integral

$$\int\limits_{\Gamma} \nabla \times \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{s} = \oint\limits_{C(\Gamma)} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l}\,. \tag{A.44}$$

Finally, we want to define the integration by parts and its extension to Green's integral formula. Let $\Omega \subset \mathcal{R}^n$, $n = 2, 3$ be a domain with smooth boundary Γ. Then, for any $u, v \in H^1(\Omega)$ the following relation holds (definition of functional spaces, e.g., H^1 see [19])

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_i} v \, d\Omega = \int_{\Gamma} uv \, \boldsymbol{n} \cdot \boldsymbol{e}_i \, ds - \int\limits_{\Omega} u \frac{\partial v}{\partial x_i} \, d\Omega\,. \tag{A.45}$$

In (A.45) $\boldsymbol{n}$ denotes the outer normal of the considered domain Ω with boundary Γ. By a multiple application of (A.45), we arrive at **Green's formula**

$$\int\limits_{\Omega} \triangle u \, v \, d\Omega = \int\limits_{\Gamma} \frac{\partial u}{\partial \boldsymbol{n}} v \, ds - \int\limits_{\Omega} \nabla u \cdot \nabla v \, d\Omega \tag{A.46}$$

for all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$.

B. Tensors and Index Notation

Tensors are simply speaking a linear mapping. E.g., a second order tensor $[\boldsymbol{A}]$ is a linear mapping that associates a given vector $\boldsymbol{u}$ with a second vector $\boldsymbol{v}$ by

$$\boldsymbol{v} = [\boldsymbol{S}]\boldsymbol{u}\,.$$

The term linear in the above relation implies that given two arbitrary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and two arbitrary scalars α, β, then the following relation holds

$$[\boldsymbol{S}]\,(\alpha\boldsymbol{u} + \beta\boldsymbol{v}) = \alpha[\boldsymbol{S}]\boldsymbol{u} + \beta[\boldsymbol{S}]\boldsymbol{v}\,.$$

The extension to tensors of higher rank is straight forward. E.g., Hook's law maps the mechanical strain tensor $[\boldsymbol{S}]$ by the 4th order elasticity tensor $[\boldsymbol{c}]$ to the mechanical stress tensor $[\boldsymbol{\sigma}]$

$$[\boldsymbol{\sigma}] = [\boldsymbol{c}]\,[\boldsymbol{S}]\,.$$

Now, index notation is a powerful tool to write complex operations of vectors and tensors in a more readable way. However, there are times when the more conventional vector notation is more useful. It is therefore important to be able to easily convert back and forth between the two notations. Table B.1 describes our notation[1]. An index can be a *free* or a *dummy* index. For free indices, the following rules are defined:

Table B.1.: Vector and index notation.

	Vector	Index	Rank
Scalar	ξ	ξ	0
Vector	$\boldsymbol{u}$	u_i	1
Tensor (2nd order)	$[\boldsymbol{A}]$	A_{ij}	2
Tensor (3nd order)	$[\boldsymbol{B}]$	B_{ijk}	3
Tensor (4nd order)	$[\boldsymbol{C}]$	C_{ijkl}	4

- The number of free indices equals the rank as displayed in Tab. B.1. Thereby, a scalar is a tensor with rank 0, and a vector is a tensor of rank 1. Tensors may assume a rank of any integer greater than or equal to zero. Please note that it is just allowed to sum together tensors with equal rank.

[1] Our notation does not differ between tensors of different orders.

- A free index appears once and only once within each additive term and remains within the expression after the operation has been performed, e.g.

$$a_i = \epsilon_{ijk} b_j c_k + A_{ij} d_j \,. \tag{B.1}$$

- The same letter must be used for the free index in every additive term.

- The first free index in a term corresponds to the row, and the second corresponds to the column. Thus, a vector (which has only one free index) is written as a column of three rows

$$\boldsymbol{u} = u_i = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

  and a second order tensor as

$$[\boldsymbol{A}] = A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

A dummy index defines an index, which does not appear in the final expression any more. The rules are as follows:

- A dummy index appears twice within an additive term of an expression. For the example above (see (B.1)), the dummy indices are j and k.

- A dummy index implies a summation over the range of the index, e.g.

$$a_{ii} = a_{11} + a_{22} + a_{33} \,.$$

For many operations we use the Kronecker delta (2nd order tensor)

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B.2}$$

and the alternating unit tensor (3rd order tensor)

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 231 \text{ or } 312 \\ 0 & \text{if any two indices are the same} \\ -1 & \text{if } ijk = 132,\ 213 \text{ or } 321 \end{cases} \tag{B.3}$$

Thereby, the following relation can be explored

$$\epsilon_{ijk} = \frac{1}{2}(i-j)\,(j-k)\,(k-i)\,.$$

With these definitions, we may write vector and tensor operations using index notation. Here, we list the most useful ones:

- Scalar product of two vectors

$$\boldsymbol{a} \cdot \boldsymbol{b} = c \;\rightarrow\; a_i b_i = c \tag{B.4}$$

- Vector product of two vectors

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \;\rightarrow\; \epsilon_{ijk} a_j b_k = c_i \tag{B.5}$$

- Gradient of a scalar

$$\nabla \phi = \boldsymbol{u} \;\rightarrow\; \frac{\partial \phi}{\partial x_i} = u_i \tag{B.6}$$

- Gradient of a vector

$$\nabla \boldsymbol{a} = \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_2}{\partial x_1} & \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_1}{\partial x_2} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_3}{\partial x_2} \\ \frac{\partial a_1}{\partial x_3} & \frac{\partial a_2}{\partial x_3} & \frac{\partial a_3}{\partial x_3} \end{pmatrix} \rightarrow \frac{\partial a_i}{\partial x_j} \tag{B.7}$$

- Gradient of a second order tensor

$$\nabla [\boldsymbol{A}] = \frac{\partial [\boldsymbol{A}]}{\partial \boldsymbol{x}} = \sum_{i,j,k=1}^{3} \frac{\partial A_{ij}}{\partial x_k} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{B.8}$$

- Divergence of a vector

$$\nabla \cdot \boldsymbol{a} = b \;\rightarrow\; \frac{\partial a_i}{\partial x_i} = b \tag{B.9}$$

- Divergence of a second order tensor

$$\nabla \cdot [\boldsymbol{A}] = \sum_{i,j=1}^{3} \frac{\partial A_{ij}}{\partial x_j} \boldsymbol{e}_i \tag{B.10}$$

- Curl of a vector

$$\nabla \times \boldsymbol{a} = \boldsymbol{b} \;\rightarrow\; \epsilon_{ijk} \frac{\partial a_k}{\partial x_j} = b_i \tag{B.11}$$

- Double product or double contraction of two second order tensors

$$[\boldsymbol{A}] : [\boldsymbol{B}] = c \;\rightarrow\; A_{ij} B_{ij} = c \tag{B.12}$$

or of a fourth order tensor with a second order tensors, e.g. Hooks law

$$[\boldsymbol{\sigma}] = [\boldsymbol{c}] : [\boldsymbol{S}] \tag{B.13}$$

- Dyadic or tensor product

$$\boldsymbol{a} \otimes \boldsymbol{b} = [\boldsymbol{C}] \;\rightarrow\; a_i b_j = C_{ij} \tag{B.14}$$

$$[\boldsymbol{A}] \otimes \boldsymbol{b} = [\boldsymbol{C}] \;\rightarrow\; A_{ij} b_k = C_{ijk} \tag{B.15}$$

$$[\boldsymbol{A}] \otimes [\boldsymbol{B}] = [\boldsymbol{D}] \;\rightarrow\; A_{ij} B_{kl} = D_{ijkl} \tag{B.16}$$

- Product of two tensors

$$[\boldsymbol{A}][\boldsymbol{B}] = [\boldsymbol{C}] \;\rightarrow\; A_{ij}B_{jk} = C_{ik} \tag{B.17}$$

  Note that only the inner index is to be summed.

- Trace of a tensor

$$\mathrm{tr}\left([\boldsymbol{A}]\right) = b \;\rightarrow\; A_{ii} = b \tag{B.18}$$

C. Generalized Functions

In reality, dissipative effects cause discontinuities to be smooth and real signals to decay for $t \to \infty$. However, in idealized models, we often cannot describe the properties by ordinary functions. Two simple examples: (1) a point source is zero everywhere except in one point, where it is infinitely large; (2) the function $\sin \omega t$ is not a decaying function, which in the classical sense cannot be Fourier-transformed.

So, in general we can state that our mathematical apparatus for functions is too restricted and so it makes sense to extend it to so called *generalized functions*.

C.1. Basics

Definition (Lebesgue). A function $f(\boldsymbol{x})$ is locally integrable in R^n if

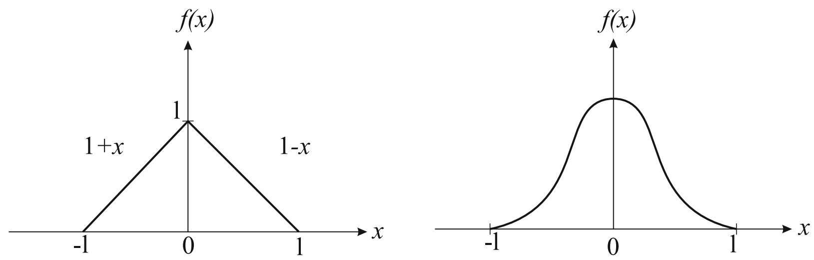
$$\int_{\Omega} |f(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}$$

exists for every bounded volume Ω in R^n. A function $f(\boldsymbol{x})$ is locally integrable on a hyper-surface in R^n if

$$\int_{\Gamma} |f(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{s}$$

exists for every bounded region Γ in R^n.

Definition. The support denoted by *supp* of a function $f(\boldsymbol{x})$ is the closure of the set of all points $\boldsymbol{x}$ such that $f(\boldsymbol{x}) \neq 0$. If $\mathit{supp}\, f$ is a bounded set, then f is said to have compact support (see Fig. C.1a).



(a) Function with compact support. (b) Infinitely differentiable functions with compact support.

Figure C.1.: Special properties of functions.

We know that the delta function $\delta(\boldsymbol{x})$ becomes meaningful, if it is first multiplied by a sufficient smooth auxiliary function and then integrated over the entire space, e.g. in the one-dimensional space

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\;\mathrm{d}x = \phi(0)\,.$$

Therefore, we will follow the **Schwartz-Sobolev** approach, which includes the following steps: (1) operators on ordinary functions such as differentiation and Fourier transform are extended by first writing these operators as *functionals* for ordinary functions; (2) extend it for all generalized functions, e.g., also for the delta function, etc. In doing so, we first define the space D of all test functions $\phi(x)$, which are infinitely differentiable functions with bounded support. The prototype of such a test function belonging to D is

$$\phi(x) = \begin{cases} e^{-\frac{a^2}{a^2-r^2}} & r < a \\ 0 & r > a \end{cases}$$

as displayed in Fig. C.1b (for details see [20]). A linear functional of f on the space D is an operation by which we assign to every test function $\phi(x)$ a real (complex) number - a functional - denoted by

$$< f, \phi > = \int_{-\infty}^{\infty} f\,\phi\;\mathrm{d}x$$

such that

$$< f, c_1\phi_1 + c_2\phi_2 > = c_1 < f, \phi_1 > + c_2 < f, \phi_2 >$$

for arbitrary test functions ϕ_1, ϕ_2 and real numbers c_1. c_2.

Definition. A linear functional on D is *continuous*, iff the sequence of numbers $< f, \phi_m >$ converges to $< f, \phi >$, when the sequence of test functions $\{\phi_m\}$ converges to the test function ϕ. Thus

$$\lim_{m\to\infty} < f, \phi_m > = < f, \lim_{m\to\infty} \phi_m > \;.$$

Definition. A continuous linear functional on the space D of test functions is called a *distribution.*

So, every locally integrable function $f(x)$ generates a distribution through the formula

$$< f, \phi > = \int_{R^n} f(x)\phi(x)\;\mathrm{d}x\,,$$

and is denoted a *regular* distribution. All other distributions are called *singular*, e.g. a distribution with the singular function δ. The space of all distributions on D is denoted by D', which is larger as D and which is also a linear space (see Fig. C.2). It forms a generalization of the class of locally integrable functions because it contains functions such as $\delta(x)$ that are not locally integrable. For this reason distributions are also called *generalized functions.* We shall use the term *distribution* and *generalized functions* interchangeably.

Example: The Heaviside distribution in R^n is

$$< H_\Omega, \phi > = \int_\Omega \phi(x)\;\mathrm{d}x \quad \text{where } H_\Omega(x) = \begin{cases} 1, & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

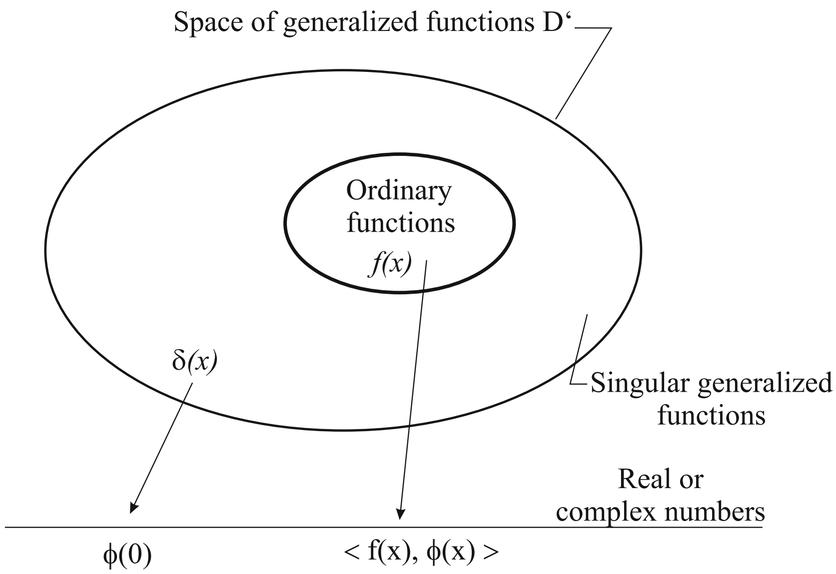
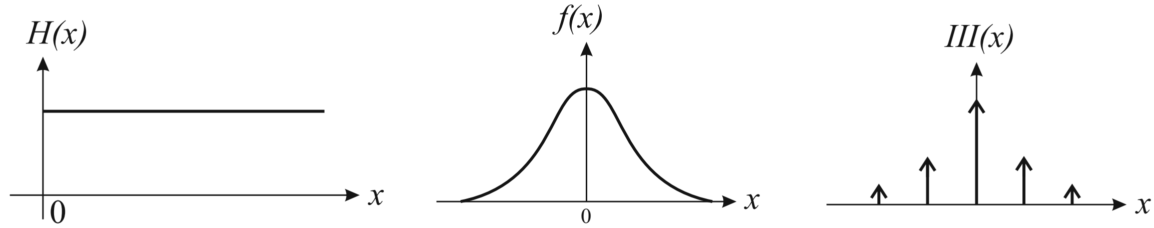


Figure C.2.: Generalized functions are continuous linear functionals on space D of test functions.



(a) Heaviside functions. (b) Sampling function with compact support.

Figure C.3.: Special functions.

For R^1 it becomes (see function in Fig. C.3a)

$$< H, \phi > = \int_0^\infty \phi(x) \, \mathrm{d}x \, .$$

Since $H(x)$ is a piecewise continuous function, this is a regular distribution.

Example: An infinite sequence of delta functions is described by

$$III(x) = \sum_{n=-\infty}^{\infty} \delta(x - n) \, .$$

This is called the sampling or replicating function (see Fig. C.3b) because it gives information about the function $f(x)$ at $x = n$

$$< III(x), f(x) > = \sum_{n=-\infty}^{\infty} f(x)\delta(x - n) \, .$$

Since the delta function is not locally integrable, this distribution is a singular distribution.

C.2. Special properties

In the following, we will derive important properties of generalized functions.

C.2.1. Shift operator

Let $f(x)$ be an ordinary function, which we shift by a value of h. Then, the linear functional on the space of test functions computes as

$$< f(x+h), \phi(x) > = \int_{-\infty}^{\infty} f(x+h)\phi(x)\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\phi(x-h)\, \mathrm{d}x\,.$$

This rule can now be used for all generalized functions in D', e.g. for the delta function $\delta(x)$

$$< \delta(x+h), \phi(x) > = \int_{-\infty}^{\infty} \delta(x+h)\phi(x)\, \mathrm{d}x = \int_{-\infty}^{\infty} \delta(x)\phi(x-h)\, \mathrm{d}x = \phi(-h)\,.$$

C.2.2. Linear change of variables

Let $< f, \phi >$ be a regular distribution generated by the function $f(x)$ that is locally integrable in R^n. Now, let $x = Ay - a$, where A is a $n \times n$ matrix with $det(A) \neq 0$ and a a constant vector, be a non-singular linear transformation of the space R^n onto itself. Then we have

$$\begin{aligned} < f(Ay-a), \phi(y) > &= \int_{R^n} f(Ay-a)\phi(y)\, \mathrm{d}y \\ &= \frac{1}{|det(A)|} \int_{R^n} f(x)\phi\left(A^{-1}(x+a)\right)\, \mathrm{d}x \\ &= \frac{1}{|det(A)|} < f(x), \phi\left(A^{-1}(x+a)\right) >\,, \end{aligned} \tag{C.1}$$

where A^{-1} is the inverse of the matrix A. In special, we have for the delta function

$$< \delta(ax), \phi(x) > = \frac{1}{|a|} < \delta(x), \phi(x) > \;. \tag{C.2}$$

So we can simply write $\delta(ax) = (1/|a|)\delta(x)$.

C.2.3. Derivatives of generalized functions

Let $f(x)$ be an ordinary function out of D, e.g. $f \in C^1$. Then we can write

$$< f'(x), \phi(x) > = \int_{-\infty}^{\infty} f'(x)\phi(x)\, \mathrm{d}x\,.$$

Performing an integration by parts results in

$$< f'(x), \phi(x) > = \underbrace{f(x)\phi(x)|_{-\infty}^{\infty}}_{=0 \text{ due to local support of } \phi(x)} - \int_{-\infty}^{\infty} f(x)\,\phi'(x)\, \mathrm{d}x\,. \tag{C.3}$$

This result can be extended to define the derivatives of all generalized functions in D'

$$< f^n(x), \phi(x) > = (-1)^n \; < f(x), \phi^n(x) > , \tag{C.4}$$

which states that generalized functions have derivatives of all orders.

Example: The derivative of the delta function $\delta'(x)$ has the property

$$< \delta'(x), \phi(x) > = -\int_{-\infty}^{\infty} \delta(x)\phi'(x) \, \mathrm{d}x = -\phi'(0) \,. \tag{C.5}$$

Example: The derivative of the Heaviside function computes as

$$\begin{aligned} < H'(x), \phi(x) > \;&=\; \int_{-\infty}^{\infty} H'(x)\phi(x) \, \mathrm{d}x \\ &=\; - < H(x), \phi'(x) > \\ &=\; \int_{0}^{\infty} \phi'(x) \, \mathrm{d}x = - \left. \phi(x) \right|_0^\infty \\ &=\; \phi(0) = < \delta(x), \phi(x) > \;. \end{aligned} \tag{C.6}$$

C.2.4. Multidimensional delta function

In multidimensional, $\delta(\boldsymbol{x})$ has a simple interpretation by

$$< \delta(\boldsymbol{x}), \phi(\boldsymbol{x}) > = \int_{-\infty}^{\infty} \delta(\boldsymbol{x})\phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \phi(\boldsymbol{0}) \,.$$

Thus,

$$\delta(\boldsymbol{x}) = \delta(x_1)\,\delta(x_2)\,\delta(x_3)...\delta(x_n) \,,$$

where $\boldsymbol{x} = (x_1, x_2, .., x_n)$. Of great interest are applications of $\delta(f)$ and $\delta'(f)$, where $f = 0$ is a surface in the three-dimensional space as displayed in Fig. C.4. Then, for a test function $\phi(\boldsymbol{x})$ defined in Ω and on Γ we have the following properties

$$\int_{-\infty}^{\infty} \phi(\boldsymbol{x}) \nabla H(f) \, \mathrm{d}\boldsymbol{x} \;=\; \oint_\Gamma \phi(\boldsymbol{x})\boldsymbol{n} \, \mathrm{d}s = \oint_\Gamma \phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{s} \tag{C.7}$$

$$\int_{-\infty}^{\infty} \phi(\boldsymbol{x}) \frac{\partial H(f)}{\partial x_j} \, \mathrm{d}\boldsymbol{x} \;=\; \oint_\Gamma \phi(\boldsymbol{x}) n_j \, \mathrm{d}s = \oint_\Gamma \phi(\boldsymbol{x}) \, \mathrm{d}s_j \,. \tag{C.8}$$

In the following, we want to proof these properties. First of all, we may write by the chain rule of differentiation

$$\frac{\partial H(f)}{\partial x_j} = \underbrace{\frac{\partial H(f)}{\partial f}}_{=\delta(f)} \frac{\partial f}{\partial x_j} \;\Rightarrow\; \nabla H(f) = \delta(f)\,\nabla f \,.$$

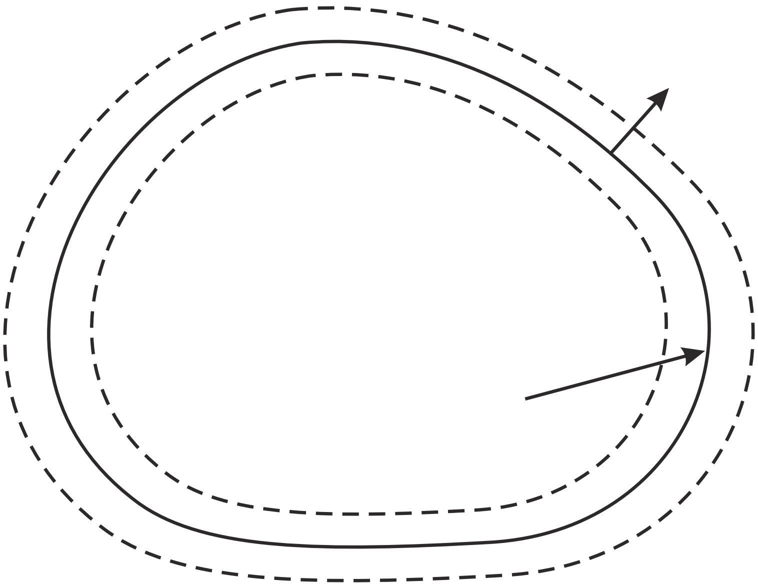


Figure C.4.: Surface defined by $f = 0$.

Thereby, the gradient of f points in the direction of $\boldsymbol{n}$. In a next step, we will decompose the volume element into a surface element $\mathrm{d}s$ and a line element $\mathrm{d}l_\perp$ in the direction of $\boldsymbol{n}$, and write

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}l_\perp \; \mathrm{d}s \,.$$

Since, f is zero on the surface Γ, a Taylor expansion up to first order results in

$$f = \left(\frac{\partial f}{\partial l_\perp}\right)_\Gamma l_\perp = |\nabla f| \, l_\perp \,.$$

Using this relation and the property (C.2), we may write

$$\delta(f) = \delta(|\nabla f| l_\perp) = \frac{\delta(l_\perp)}{|\nabla f|} \,.$$

Hence, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(\boldsymbol{x}) \nabla H(f) \, \mathrm{d}\boldsymbol{x} &= \int_{-\infty}^{\infty} \phi(\boldsymbol{x}) \nabla f \, \delta(f) \, \mathrm{d}\boldsymbol{x} \\
&= \int_{-\infty}^{\infty} \phi(\boldsymbol{x}) \underbrace{\frac{\nabla f}{|\nabla f|}}_{\boldsymbol{n}} \delta(l_\perp) \, \mathrm{d}l_\perp \, \mathrm{d}s \\
&= \oint_\Gamma \phi(\boldsymbol{x}) \boldsymbol{n} \, \mathrm{d}s \,, \qquad \text{(C.9)}
\end{aligned}$$

since $\boldsymbol{n} = \nabla f / |\nabla f|$.

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