Problem Sheet

1.1 First order linear equations;

In each of Problems 1 through 8 find the solution of the given initial value problem.

1. $y' - y = 2te^{2t},\ \ y(0) = 1$
2. $y' + 3y = te^{-3t},\ \ y(1) = 0$
3. $ty' + 2y = t^2 - t + 1,\ \ y(1) = 1/2,\ \ t > 0$
4. $y' + (2/t)y = \cos t\,/t^2,\ \ y(\pi) = 0,\ \ t > 0$
5. $y' - 4y = e^{4t},\ \ y(0) = 2$
6. $ty' + 2y = \sin t,\ \ y(\pi/2) = 1,\ \ t > 0$
7. $t^3y' + 4t^2y = e^{-t},\ \ y(-1) = 0,\ \ t < 0$
8. $ty' + (t+1)y = t,\ \ y(\ln 2) = 1,\ \ t > 0$

In each of Problems 9 and 10:

(a) Let a_0 be the value of a for which the transition from one type of behaviour to another occurs. Solve the initial value problem and find the critical value a_0 exactly.

(b) Describe the behaviour of the solution corresponding to the initial value a_0.

9. $y' - y/2 = 2\cos t\,,\ \ y(0) = a$
10. $2y' - y = e^{t/3},\ \ y(0) = a$

11. *Consider the initial value problem:

$$y' + y/2 = 2\cos t\,,\ \ y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

12. *Consider the initial value problem:

$$y' + 2y/3 = 1 - t/2,\ \ y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the $t-$ axis.

1.2 Separable equations

In each of Problems 1 through 8 solve the given differential equation.

1. $y' = x^2/y$
2. $y' + y^2\sin x = 0$
3. $y' = x^2/y(1 + x^3)$
4. $y' = (3x^2 - 1)/(3 + 2y)$
5. $y' = (\cos^2 x)(\cos^2 2y)$
6. $xy' = (1 - y^2)^{1/2}$
7. $y' = (x - e^{-x})/(y + e^y)$
8. $y' = x^3/(1 + y^2)$

9. *Solve the initial value problem and determine the interval in which the solution is valid.

$$y' = (1 + 3x^2)/(3y^2 - 6y),\ \ y(0) = 1.$$

10. *Solve the initial value problem and determine the interval in which the solution is valid.

$$y' = 3x^2/(3y^2 - 4),\ \ y(1) = 0.$$

In each of Problems 11 through 14 solve the given differential equation.

11. $y' = (x^2 + xy + y^2)/x^2$
12. $y' = (x^2 + 3y^2)/2xy$
13. $y' = (4y - 3x)/(2x - y)$
14. $y' = -(4x + 3y)/(2x + y)$

1.3 Modelling with first order equations

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains $150\ L$ of a dye solution with a concentration of $1\ g/L$. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of $2\ L/min$, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

2. A tank initially contains $120\ L$ of pure water. A mixture containing a concentration of $\gamma\ g/L$ of salt enters the tank at a rate of $2\ L/min$, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

3. A tank originally contains $100\ gal$ of fresh water. Then water containing $12\ lb$ of salt per gallon is poured into the tank at a rate of $2\ gal/min$, and the mixture is allowed to leave at the same rate. After $10\ min$ the process is stopped, and fresh water is poured into the tank at a rate of $2\ gal/min$, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional $10\ min$.

4. A tank with a capacity of $500\ gal$ originally contains $200\ gal$ of water with $100\ lb$ of salt in solution. Water containing $1\ lb$ of salt per gallon is entering at a rate of $3\ gal/min$, and the mixture is allowed to flow out of the tank at a rate of $2\ gal/min$. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

5. *A tank contains $100\ gallons$ of water and $50\ oz$ of salt.Water containing a salt concentration of $1/4\ (1\ + 1/2 \sin t)\ oz/gal$ flows into the tank at a rate of $3\ gal/min$, and the mixture in the tank flows out at the same rate.
   (a) Find the amount of salt at any time?
   (b) Plot the solution for a time period long enough so that you see the ultimate behaviour of the graph.
   (c) The long-time behaviour of the solution is an oscillation about a certain constant level. What is that level? What is the amplitude of the oscillation?

6. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.
   (a) Find the time T required for the original sum to double in value as a function of r.
   (b) Determine T if $r = 8\%$.
   (c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

7. A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.
   (a) Determine the sum $S(t)$ accumulated at any time t.
   (b) If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.
   (c) If $k = \$2000/year$, determine the return rate r that must be obtained to have \$1 million available in 40 years.

8. A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

9. A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.
   (a) Determine the maximum amount that this buyer can afford to borrow.
   (b) Determine the total interest paid during the term of the mortgage.

10. *A recent college graduate borrows \$100,000 at an interest rate of 9% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800(1 + t/120)$, where t is the number of months since the loan was made.
   (a) Assuming that this payment schedule can be maintained, when will the load be fully paid?
   (b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

1.4 Population dynamics

Problems 1 through 6 involve equations of the form $dy/d = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable, or semistable. Draw the phase line, and sketch several graphs of solutions in the ty-plane.

1. $y' = -k(y-2)^2,\ \ k > 0,\ \ -\infty < y_0 < \infty$
2. $y' = y^2(y^2-1),\ \ -\infty < y_0 < \infty$
3. $y' = y(4-y^2),\ \ -\infty < y_0 < \infty$
4. $y' = ay - b\sqrt{y},\ \ a > 0,\ \ b > 0,\ \ y_0 \geq 0$
5. $y' = y^2(1-y)^2,\ \ -\infty < y_0 < \infty$
6. $y' = y^2(4-y^2),\ \ -\infty < y_0 < \infty$

7. Suppose that a certain population obeys the logistic equation $y' = ry[1 - (y/K)]$.
   (a) If $y_0 = K/4$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ per year.
   (b) If $y_0/K = \alpha$, find the time T at which $y(T)/K = \beta$, where $0 < \alpha, \beta < 1$. Observe that $T \to \infty$ as $\alpha \to 0$ or as $\beta \to 1$. Find the value of T for $r = 0.025$ per year, $\alpha = 0.1$ and $\beta = 0.9$.
8. Another equation that has been used to model population growth is the Gompertz equation

$$y' = ry \ln(K/y),$$

   where r and K are positive constants.
   (a) Sketch the graph of $f(y)$ versus y, find the critical points, and determine whether each is asymptotically stable or unstable.
   (b) For $0 \leq y \leq K$ determine where the graph of y versus t is concave up and where it is concave down.

9. At a given level of effort, it is reasonable to assume that the rate at which the fish are caught depends on the population y: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by Ey, where E is a positive constant, with units of $1/time$, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$y' = r(1 - y/K)y - Ey.$$

   This equation is known as the Schaefer model after the biologist M. B. Schaefer, who applied it to fish populations.
   (a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$.
   (b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.
   (c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Find Y as a function of the effort E; the graph of this function is known as the yield-effort curve.
   (d) Determine E so as to maximize Y and thereby find the maximum sustainable yield Y_m.

10. In this problem we assume that fish are caught at a constant rate h independent of the size of the fish population. Then y satisfies

$$y' = r(1 - y/K)y - h.$$

   The assumption of a constant catch rate h may be reasonable when y is large but become less so when y is small.
   (a) If $h < rK/4$, show that the equation has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.
   (b) Show that y_1 is unstable and y_2 is asymptotically stable.
   (c) From a plot of $f(y)$ versus y, show that if the initial population $y_0 > y_1$, then $y \to y_2$ as $t \to \infty$, but that if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.
   (d) If $h > rK/4$, show that y decreases to zero as t increases regardless of the value of y_0.

(e) If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable. Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 9(d). The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

1.5 Difference between lin. and nonlin. equations

In each of Problems 1 through 4 determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t-3)y' + (\ln t)y = 2t,\ \ y(1) = 2$
2. $t(t-5)y' + y = 0,\ \ y(2) = 1$
3. $y' + (\tan t)y = \sin t,\ \ y(\pi) = 0$
4. $(4-t^2)y' + 2ty = 3t^2,\ \ y(-3) = 1$

In each of Problems 5 through 8 state the region in the ty-plane where the solution is certain to exist and unique.

5. $y' = (t-y)/(2t+5y)$
6. $y' = (1-t^2-y^2)^{1/2}$
7. $y' = (t^2+y^2)^{3/2}$
8. $y' = \ln|ty|/(1-t^2+y^2)$

In each of Problems 9 through 10 solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

9. $y' = -4t/y,\ \ y(0) = y_0$
10. $y' = 2ty^2,\ \ y(0) = y_0$

11. Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem
$$y' = (-t + (t^2+4y)^{1/2})/2\,,\ \ y(2) = -1.$$
Where are these solutions valid?
   (a) Explain why the existence of two solutions of the given problem does not contradict the uniqueness Theorem.
   (b) Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y2(t)$.

In each of Problem 12 through 13 solve the given differential equation.

12. $t^2y' + 2ty - y^3 = 0,\ \ t > 0$
13. $y' = ry - ky^2,\ \ r > 0,\ \ k > 0$

1.6 Exact equation

Determine whether or not each of the equations in Problems 1 through 6 is exact. If it is exact, find the solution.

1. $(2x + 4y) + (2x - 2y)y' = 0$
2. $(2x + 3) + (2y - 2)y' = 0$
3. $(2x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$
4. $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$
5. $y' = -(ax + by)/(bx + cy)$
6. $y' = -(ax - by)/(bx - cy)$

In each of Problems 7 and 8 find the value of b for which the given equation is exact and then solve it using that value of b.

7. $(xy^2 + bx^2y) + (x + y)x^2y' = 0$
8. $(ye^{2xy} + 4x^3) + bxe^{2xy}y' = 0$

In each of Problems 9 through 14 show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

9. $(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$
10. $y' = e^{2x} + y - 1$
11. $1 + (x/y - \sin y)y' = 0$
12. $y + (2xy - e^{-2y})y' = 0$
13. $e^x + (e^x \cot y + 2y \csc y)y' = 0$
14. $(4x^3/y^2 + 3/y) + (3x/y^2 + 4y)y' = 0$

1.7 The existence and uniqueness

In each of Problem 1 through 6 let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem.

1. $y' = 2(y + 1), \;\; y(0) = 0$
2. $y' = -y - 1, \;\; y(0) = 0$
3. $y' = y + 1 - t, \;\; y(0) = 0$
4. $y' = -y/2 + t, \;\; y(0) = 0$
5. $y' = ty + 1, \;\; y(0) = 0$
6. $y' = t^2y - t, \;\; y(0) = 0$

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7. If $\partial f/\partial y$ is continuous in the rectangle D, show that there is a positive constant K such that
$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|,$$
where (t, y_1) and (t, y_2) are any two points in D having the same t coordinates. This inequality is known as a Lipshitz condition.

8. If $\phi_{n-1}(t)$ and $\phi_n(t)$ are members of the sequence $\{\phi_n(t)\}$, use the result of Problem 7 to show that
$$|f[t, \phi_n(t)] - f[t, \phi_{n-1}(t)]| \leq K|\phi_n(t) - \phi_{n-1}(t)|,$$

9. Show that if $|t| \leq h$, then

$$|\phi_1(t)| \leq M|t|,$$

where M is chosen so that $|f(t,y)| \leq M$ for (t,y) in D. Use the results of Problem 8 and the first part of Problem 9 to show that

$$|\phi_2(t) - \phi_1(t)| \leq MK|t|^2/2.$$

Show, by mathematical introduction, that

$$|\phi_n(t) - \phi_{n-1}(t)| \leq MK^{n-1}|t|^n/n! \leq MK^{n-1}h^n/n!$$

10. Note that

$$\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)]$$

Show that

$$|\phi_n(t)| \leq |\phi_1(t)| + |\phi_2(t) - \phi_1(t)| + \cdots + |\phi_n(t) - \phi_{n-1}(t)|.$$

Use the results of Problem 9 to show that

$$|\phi_n(t)| \leq \frac{M}{K}\left[Kh + \frac{(Kh)^2}{2!} + \cdots + \frac{(Kh)^n}{n!}\right].$$

Show that the sum in the second part converges as $n \to \infty$ and, hence, the sum in the first part also converges as $n \to \infty$. Conclude therefore that the sequence $\{\phi_n(t)\}$ converges since it is the sequence of partial sums of a convergent infinite series.

11. In this Problem we deal with the question of uniqueness of the solution of the integral equation

$$\phi(t) = \int_0^t f[s, \phi(s)]\, ds.$$

Suppose that ϕ and ψ are two solutions of the integral equation. Show that, for $t \geq 0$,

$$\phi(t) - \psi(t) = \int_0^t \{f[s, \phi(s)] - f[s, \psi(s)]\} ds.$$

Show that

$$|\phi(t) - \psi(t)| \leq \int_0^t |f[s, \phi(s)] - f[s, \psi(s)]| ds.$$

Use the result of Problem 7 to show that

$$|\phi(t) - \psi(t)| \leq K \int_0^t |\phi(s) - \psi(s)| ds,$$

where K is an upper bound for $|\partial f/\partial y|$ in D.

1.8 First order difference equations

In each of Problem 1 through 6 solve the given difference equation in terms of the initial value y_0. Describe the behaviour of the solution as $n \to \infty$.

1. $y_{n+1} = -0.8y_n$
2. $y_{n+1} = (n+1)/(n+2)y_n$
3. $y_{n+1} = \sqrt{(n+3)/(n+1)}y_n$
4. $y_{n+1} = (-1)^{n+1}y_n$
5. $y_{n+1} = 0.5y_n + 6$
6. $y_{n+1} = -0.5y_n + 6$

7. An investor deposit \$1000 in an account paying interest at rate of 7% compounded monthly, and also makes additional deposit of \$25 per month. Find the balance in the account after 4 years.

8. A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 12%. What monthly payment rate is required to pay off the loan in 3 years?

9. A homebuyer wishes to take out a mortgage of \$100.000 for a 30-year period. What monthly payment is required if the interest rate is 9%, 10% and 12%?

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Problem 10 through 15 deal with the difference equation $u_{n+1} = \rho u_n(1 - u_n)$.

10. Carry out the details in the linear stability analysis of the equilibrium solutions.

11. For $\rho = 3.2$ plot or calculate the solution of the equation for several initial conditions, say, $u = 0.2, 0.4, 0.6,$ and 0.8. Observe that in each case the solution approaches a steady oscillation between the same two values. This illustrates that the long-term behaviour of the solution is independent of the initial conditions. Make similar calculations and verify that the nature of the solution for large n is independent of the initial condition for other values of ρ, such as 2.6, 2.8 and 3.4.

12. Assume that $\rho > 1$. Draw a qualitative correct stairstep diagram and thereby show that if $u_0 < 0$, then $u_n \to -\infty$ as $n \to \infty$. In a similar way, determine what happens as $n \to \infty$ if $u_0 > 1$.

13. The solutions change from convergent sequences to periodic oscillations of period two as the parameter ρ passes through the value 3. To see more clearly how this happens, carry out the following calculations. Plot or calculate the solution for $\rho = 2.9, 2.95,$ and 2.99, respectively, using an initial value u_0 of your choice in the interval (0,1). In each case estimate how many iterations are required for the solution to get "very close" to the limiting value. Use any convenient interpretation of what "very close" means in the preceding sentence. Plot or calculate the solution for $\rho = 3.01, 3.05,$ and 3.1, respectively, using the same initial condition. In each case estimate how many iterations are needed to reach a steady-state oscillation. Also find or estimate the two values in the steady-state oscillation.

14. By calculating or plotting the solution for different values of ρ, estimate the value of ρ for which the solution changes from an oscillation of period 2 to one of period 4. In the same way estimate the value of ρ for which the solution changes from period 4 to period 8.

15. Let ρ_k be the value of ρ for which the solution changes from period 2^{k-1} to period 2^k. Thus, as noted in the text, $\rho_1 = 3$, $\rho_2 = 3.449$ and $\rho_3 = 3.544$. Using this values of ρ_1, ρ_2 and ρ_3, or those you found in Problem 14, calculate $(\rho_2 - \rho_1)/(\rho_3 - \rho_2)$. Let $\delta_n = (\rho_n - \rho_{n-1})/(\rho_{n+1} - \rho_n)$. It has been shown that δ_n approaches a limit δ as $n \to \infty$, where $\delta \cong 4.6692$ is known as Feigenbaum number. Determine the percentage difference between the limiting value δ and δ_2. Assume that $\delta_3 = \delta$ and use this relation to estimate ρ_4, the value of ρ at which solutions of period 16 appear.

By plotting or calculating solutions near the value of ρ_4, try to detect the appearance of a period 16 solution. Observe that

$$\rho_n = \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + \cdots + (\rho_n - \rho_{n-1}).$$

Assuming that $(\rho_4 - \rho_3) = (\rho_3 - \rho_2)\delta^{-1}, (\rho_5 - \rho_4) = (\rho_3 - \rho_2)\delta^{-2}$, and so forth, express ρ_n as a geometric sum. Then find the limit of ρ_n as $n \to \infty$. This is an estimate of the value of ρ at which the onset of chaos occurs.

2.1 Real roots

In each of Problems 1 through 4 find the general solution of the given differential equation.

1. $y'' + 3y' - 4y = 0$
2. $2y'' - 3y' + y = 0$
3. $6y'' - y' - y = 0$
4. $y'' + 3y' + 2y = 0$

In each of Problems 5 through 8 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behaviour as t increases.

5. $y'' - 3y' + 2y = 0,\ \ y(0) = 1,\ \ y'(0) = 1$
6. $y'' + 4y' + 3y = 0,\ \ y(0) = 2,\ \ y'(0) = -1$
7. $y'' + 5y' + 3y = 0,\ \ y(0) = 1,\ \ y'(0) = 0$
8. $y'' + 3y' = 0,\ \ y(0) = -2,\ \ y'(0) = 3$

9. Find the solution of the initial value problem
$$2y'' - 3y' + y = 0,\ \ y(0) = 2,\ \ y'(0) = 1/2.$$
Then determine the maximum value of the solution and also find the point where the solution is zero.

10. Solve the initial value problem $y'' - y' - 2y = 0,\ \ y(0) = \alpha,\ \ y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.

In each of Problems 11 and 12 determine the value of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

11. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$
12. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$

13. Consider the equation $ay'' + by' + cy = d$, where a, b, c and d are constants.
    (a) Find all equilibrium, or constant solutions of this differential equation.
    (b) Let y_e denote an equilibrium solution, and let $Y = y - y_e$. Thus Y is the deviation of a solution y from an equilibrium solution. Find the differential equation satisfied by Y.

14. Consider the equation $ay'' + by' + cy = 0$, where a, b and c are constants with $a > 0$. Find conditions on a, b and c such that the roots of the characteristic equation are:
    (a) real, different, and negative.
    (b) real with opposite signs.
    (c) real, different, and positive

2.2 Complex roots

In each of Problems 1 through 4 find the general solution of the given differential equation.

1. $y'' - 2y' + 2y = 0$
2. $y'' + 2y' + 2y = 0$
3. $y'' + 2y' - 8y = 0$
4. $y'' - 2y' + 6y = 0$

In each of Problems 5 through 8 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behaviour as t increases.

5. $y'' + 4y = 0,\ \ y(0) = 0,\ \ y'(0) = 1$
6. $y'' + 4y' + 5y = 0,\ \ y(0) = 1,\ \ y'(0) = 0$
7. $y'' - 6y' + 13y = 0,\ \ y(\pi/2) = 1,\ \ y'(\pi/2) = 2$
8. $y'' + y = 0,\ \ y(\pi/3) = 2,\ \ y'(\pi/3) = -4$

9. Consider the initial value problem
$$3u'' - u' + 2u = 0,\ \ u(0) = 2,\ \ u'(0) = 0.$$
   (a) Find the solution $u(t)$ of this problem.
   (b) *For $t > 0$ find the first time at which $|u(t)| = 10$.

10. Consider the initial value problem
$$5u'' + 2u' + 7u = 0,\ \ u(0) = 2,\ \ u'(0) = 1.$$
   (a) Find the solution $u(t)$ of this problem.
   (b) *Find the smallest T such that $|u(t)| \leq 0.1$ for all $t > T$.

In each of Problems 11 through 14 solve the given equation for $t > 0$.

11. $t^2y'' + ty' + y = 0$
12. $t^2y'' + 4ty' + 2y = 0$
13. $t^2y'' - 4ty' - 6y = 0$
14. $t^2y'' + 3ty' + 1.25y = 0$

In each of Problems 15 through 17 try to transform the given equation into one with constant coefficients. If it is possible, find the general solution of the given equation.

15. $y'' + ty' + e^{-t^2}y = 0,\ -\infty < t < \infty$
16. $ty'' + (t^2 - 1)y' + t^3y = 0,\ 0 < t < \infty$
17. $y'' + 3ty' + t^2y = 0,\ \ -\infty < t < \infty$

2.3 Repeated roots; Reduction of order

In each of Problems 1 through 4 find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $4y'' + 12y' + 9y = 0$
4. $4y'' - 4y' - 3y = 0$

In each of Problems 5 through 8 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behaviour as t increases.

5. $9y'' + 6y' + 82y = 0,\ \ y(0) = -1,\ \ y'(0) = 2$
6. $y'' - 6y' + 9y = 0,\ \ y(0) = 0,\ \ y'(0) = 2$
7. $9y'' - 12y' + 4y = 0,\ \ y(0) = 2,\ \ y'(0) = -1$
8. $y'' + 4y' + 4y = 0,\ \ y(-1) = 2,\ \ y'(-1) = 1$

9. Consider the initial value problem
$$4y'' + 12y' + 9y = 0,\ \ y(0) = 1,\ \ y'(0) = -4.$$
   (a) Solve the initial value problem and plot its solution for $0 \le t \le 5$.
   (b) Determine where the solution has the value zero.
   (c) Determine the coordinates (t_0, y_0) of the minimum point.
   (d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

10. Consider the initial value problem
$$4y'' + 4y' + y = 0,\ \ y(0) = 1,\ \ y'(0) = 2.$$
   (a) Solve the initial value problem and plot the solution.
   (b) Determine the coordinates (t_M, y_M) of the maximum point.
   (c) Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b.
   (d) Find the coordinates (t_M, y_M) of the maximum point in terms of b. Describe the dependency of t_M and y_M on b as b increases.

11. Consider the initial value problem
$$16y'' + 24y' + 9y = 0,\ \ y(0) = a > 0,\ \ y'(0) = -1.$$
   (a) Solve the initial value problem.
   (b) Find the critical value of a that separates solutions that become negative from those that are always positive.

In each of Problems 12 through 15 use the method of reduction of order to find a second solution of the given differential equation.

12. $t^2y'' + 2ty' - 2y = 0,\ \ t > 0,\ \ y_1 = t$
13. $t^2y'' - 4ty' + 6y = 0,\ \ t > 0,\ \ y_1 = t^2$
14. $t^2y'' + 3ty' + y = 0,\ \ t > 0,\ \ y_1 = t^{-1}$
15. $t^2y'' - t(t+2)y' + (t+2)y = 0,\ \ t > 0,\ \ y_1 = t$

2.4 Method of undetermined coefficients

In each of Problems 1 through 10 find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' + 2y' + 5y = 3\cos 2t$
3. $y'' - 2y' - 3y = 2 - 3te^{-t}$
4. $y'' + 2y' = 3 + 4\sin 2t$
5. $y'' + 9y = t^2e^{3t} + 6$
6. $y'' + 2y' + y = 2e^{-t}$
7. $2y'' + 3y' + y = t^3 + 3\sin t$
8. $y'' + y = 3\sin 2t + t\cos 2t$

9. $u'' + \omega_0^2 u = \cos \omega t, \quad \omega^2 \neq \omega_0^2$
10. $u'' + \omega_0^2 u = \cos \omega_0 t$

In each of Problems 11 through 16 find the solution of the given initial value problem.

11. $y'' + y' - 2y = 2t, \;\; y(0) = 0, \;\; y'(0) = 1$
12. $y'' - 2y' + y = t\,e^t + 4, \;\; y(0) = 1, \;\; y'(0) = 1$
13. $y'' + 4y = t^2 + 3e^t, \;\; y(0) = 0, \;\; y'(0) = 2$
14. $y'' - 2y' - 3y = 3te^t, \;\; y(0) = 1, \;\; y'(0) = 0$
15. $y'' + 4y = 2\sin 2t\,, \;\; y(0) = 2, \;\; y'(0) = -1$
16. $y'' + 2y' + 5y = 4e^{-t}\cos 2t\,, \;\; y(0) = 1, \;\; y'(0) = 0$

17. Consider the equation
$$y'' - 3y' - 4y = 2e^{-t}.$$
$y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$ are the solutions of the corresponding homogeneous equation. Adapting the method of reduction of order, seek a solution of the nonhomogeneous equation of the form $Y(t) = v(t)y_1(t) = v(t)e^{-t}$, where $v(t)$ is to be determined.
(a) Substitute $Y(t)$, $Y'(t)$ and $Y''(t)$ into the equation and show that $v(t)$ must satisfy $v'' - 5v' = 2$.
(b) Let $w(t) = v'(t)$ and show that $w(t)$ must satisfy $w' - 5w = 2$. Solve this equation for $w(t)$.
(c) Integrate $w(t)$ to find $v(t)$ and then show that
$$Y(t) = -\frac{2}{5}te^{-t} + \frac{1}{5}c_1e^{4t} + c_2e^{-t}.$$
The first term on the right side is the desired particular solution of the nonhomogeneous equation. Note that it is a product of t and e^{-t}.

2.5 Variation of parameters

In each of Problems 1 through 4 use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

1. $y'' - 5y' + 6y = 2e^t$
2. $y'' - y' - 2y = 2e^{-t}$
3. $y'' + 2y' + y = 4e^{-t}$
4. $4y'' - 4y' + y = 16e^{t/2}$

In each of Problems 5 through 12 find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

5. $y'' + y = \tan t\,, \;\; 0 < t < \pi/2$
6. $y'' + 4y = 3\csc 2t\,, \;\; 0 < t < \pi/2$
7. $y'' + 4y' + 4y = t^{-2}e^{-2t}, \;\; t > 0$
8. $y'' + 9y = 9\sec^2 3t\,, \;\; 0 < t < \pi/6$
9. $4y'' + y = 2\sec(t/2)\,, \;\; -\pi < t < \pi$
10. $y'' - 2y' + y = e^t/(1 + t^2)$
11. $y'' - 5y' + 6y = g(t)$
12. $y'' + 9y = g(t)$

In each of Problems 13 through 16 verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

13. $t^2y'' - 2y = 3t^2 - 1,\ \ t > 0,\ \ y_1(t) = t^2,\ \ y_2(t) = t^{-1}$
14. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3,\ \ t > 0,\ \ y_1(t) = t,\ \ y_2(t) = te^t$
15. $(1-t)y'' + ty' - y = 2(t-1)^2e^{-t},\ \ 0 < t < 1,\ \ y_1(t) = e^t,\ \ y_2(t) = t$
16. $ty'' - (1+t)y' + y = t^2e^{2t},\ \ t > 0,\ \ y_1(t) = 1 + t,\ \ y_2(t) = e^t$

3.1 Real eigenvalues

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behaviour of the solution as $t \to \infty$.

(b) Sketch a phase portrait of the system.

1. $\boldsymbol{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x}$
2. $\boldsymbol{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \boldsymbol{x}$
3. $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \boldsymbol{x}$
4. $\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x}$
5. $\boldsymbol{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \boldsymbol{x}$
6. $\boldsymbol{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \boldsymbol{x}$

In each of Problems 7 through 10 find the general solution of the given system of equations.

7. $\boldsymbol{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \boldsymbol{x}$
8. $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \boldsymbol{x}$
9. $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \boldsymbol{x}$
10. $\boldsymbol{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \boldsymbol{x}$

In each of Problem 11 and 12 solve the given initial value problem. Describe the behaviour of the solution as $t \to \infty$.

11. $\boldsymbol{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \boldsymbol{x}, \ \ \boldsymbol{x}(0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$
12. $\boldsymbol{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \boldsymbol{x}, \ \ \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Solve the given system of equations in each of Problem 13 and 14. Assume that $t > 0$.

13. $t\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x}$
14. $t\boldsymbol{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \boldsymbol{x}$

15. Consider the equation
$$ay'' + by' + cy = 0,$$
where a, b and c are constants with $a \neq 0$. The general solution depend on the roots of the characteristic equation
$$ar^2 + br + c = 0.$$
(a) Transform the equation into a system of first order equations by letting $x_1 = y$, $x_2 = y'$. Find the system of equations $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ satisfied by $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.
(b) Find the equation that determines the eigenvalues of the coefficient matrix $\boldsymbol{A}$ in part (a). Note that this equation is just the characteristic equation of the original equation.

16. Consider the system
$$\boldsymbol{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \boldsymbol{x}.$$
(a) Solve the system for $\alpha = 0.5$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.
(b) Solve the system for $\alpha = 2$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.
(c) In parts (a) and (b) solutions of the systems exhibit two quite different types of behaviour. Find the eigenvalues of the coefficient matrix in terms of α and determine the value of α between 0.5 and 2 where the transition from one type of behaviour to the other occurs.

3.2 Complex eigenvalues

In each of Problems 1 through 6:
(a) Express the general solution of the given system of equations in terms of real-valued functions.
(b) Also sketch a few of the trajectories and describe the behaviour of the solution as $t \to \infty$.

1. $\boldsymbol{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \boldsymbol{x}$
2. $\boldsymbol{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \boldsymbol{x}$
3. $\boldsymbol{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \boldsymbol{x}$
4. $\boldsymbol{x}' = \begin{pmatrix} 2 & -5/2 \\ 9/5 & -1 \end{pmatrix} \boldsymbol{x}$
5. $\boldsymbol{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \boldsymbol{x}$
6. $\boldsymbol{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \boldsymbol{x}$

In each of Problems 7 through 8 express the general solution of the given system of equations in terms of real-valued functions.

7. $\boldsymbol{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \boldsymbol{x}$
8. $\boldsymbol{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \boldsymbol{x}$

In each of Problem 9 and 12 the coefficient matrix contains a parameter α. In each of these problems:

(a) Determine the eigenvalues in terms of α.
(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.
(c) Sketch a phase portrait for a value of α slightly below, and for another value slightly above, each critical value.

9. $\boldsymbol{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \boldsymbol{x}$
10. $\boldsymbol{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \boldsymbol{x}$
11. $\boldsymbol{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \boldsymbol{x}$
12. $\boldsymbol{x}' = \begin{pmatrix} 5/4 & 3/4 \\ \alpha & 5/4 \end{pmatrix} \boldsymbol{x}$

Solve the given system of equations in each of Problem 13 and 14. Assume that $t > 0$.

13. $t\boldsymbol{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \boldsymbol{x}$
14. $t\boldsymbol{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \boldsymbol{x}$

3.3 Fundamental matrices

In each of Problems 1 through 8:

(a) Find a fundamental matrix for the given system of equations.
(b) Also find the fundamental matrix $\boldsymbol{\phi}(t)$ satisfying $\boldsymbol{\phi}(0) = \boldsymbol{I}$.

1. $\boldsymbol{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x}$
2. $\boldsymbol{x}' = \begin{pmatrix} -3/4 & 1/2 \\ 1/8 & -3/4 \end{pmatrix} \boldsymbol{x}$
3. $\boldsymbol{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \boldsymbol{x}$
4. $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \boldsymbol{x}$
5. $\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x}$
6. $\boldsymbol{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \boldsymbol{x}$
7. $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \boldsymbol{x}$
8. $\boldsymbol{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \boldsymbol{x}$

9. Solve the initial value problem
$$\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
by using the fundamental matrix $\boldsymbol{\phi}(t)$ found in Problem 5.

10. Solve the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \boldsymbol{x}, \ \ \boldsymbol{x}(0) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

by using the fundamental matrix $\boldsymbol{\phi}(t)$ found in Problem 6.

3.4 Repeated eigenvalues

In each of Problems 1 through 6:

(a) Find the general solution of the system of equations.

(b) Describe the behaviour of the solutions as $t \to \infty$.

1. $\boldsymbol{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \boldsymbol{x}$
2. $\boldsymbol{x}' = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix} \boldsymbol{x}$
3. $\boldsymbol{x}' = \begin{pmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{pmatrix} \boldsymbol{x}$
4. $\boldsymbol{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \boldsymbol{x}$
5. $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \boldsymbol{x}$
6. $\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \boldsymbol{x}$

In each of Problem 7 and 8 solve the given system of equations. Assume that $t > 0$.

7. $t\boldsymbol{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \boldsymbol{x}$
8. $t\boldsymbol{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \boldsymbol{x}$

9. Show that all solutions of the system

$$\boldsymbol{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \boldsymbol{x}.$$

approaches zero as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$.

10. Consider the system

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \boldsymbol{x}.$$

(a) Show that $r = 2$ is an eigenvalues of algebraic multiplicity 3 of the coefficient matrix $\boldsymbol{A}$ and that there is only one corresponding eigenvector, namely,

$$\boldsymbol{\xi}^{(\mathbf{1})} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Using the information in part (a), write down one solution $\boldsymbol{x}^{(\mathbf{1})}(t)$ of the system. There is no other solution of the purely exponential form $\boldsymbol{x} = \boldsymbol{\xi}e^{rt}$.

(c) To find a second solution, assume that $\boldsymbol{x} = \boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t}$. Show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy the equations

$$(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{\xi} = \mathbf{0}, \ \ (\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Since $\boldsymbol{\xi}$ has already been found in part (a), solve the second equation for $\boldsymbol{\eta}$. Neglect the multiple of $\boldsymbol{\xi}^{(1)}$ that appears in $\boldsymbol{\eta}$, since it is leads only to a multiple of the first solution $\boldsymbol{x}^{(1)}$. Then write down a second solution $\boldsymbol{x}^{(2)}(t)$ of the system.

(d) To find a third solution, assume that $\boldsymbol{x} = \boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} te^{2t} + \boldsymbol{\zeta} e^{2t}$. Show that $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ satisfy the equations

$$(\boldsymbol{A} - 2\boldsymbol{I})\xi = \boldsymbol{0},\ (\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi},\ (\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{\zeta} = \boldsymbol{\eta}.$$

The first two equations are the same as in part (c), so solve the third equation for $\boldsymbol{\zeta}$, again neglecting the multiple of $\boldsymbol{\xi}^{(1)}$ that appears. Then write down a third solution $\boldsymbol{x}^{(3)}(t)$ of the system.

(e) Write down a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system.

(f) Form a matrix $\boldsymbol{T}$ with eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column and the generalized eigenvectors $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ in the second and third columns. Then find $\boldsymbol{T}^{-1}$ and form the product $\boldsymbol{J} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$. The matrx $\boldsymbol{J}$ is the Jordan form $\boldsymbol{A}$.

3.5 Nonhomogeneous linear systems

In each of Problems 1 through 4 find the general solution of the given system of equations.

1. $\boldsymbol{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$
2. $\boldsymbol{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}$
3. $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$
4. $\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$

4.1 Locally linear systems

In each of Problem 1 and 4 verify that (0,0) is a critical point, show that the system is locally linear, and discuss the type and stability of the critical point (0,0) by examining the corresponding linear system.

1. $dx/dt = -x + y + 2xy,\ \ dy/dt = -4x - y + x^2 - y^2$
2. $dx/dt = x - y^2,\ \ dy/dt = x - 2y + x^2$
3. $dx/dt = (1 + x)\sin y,\ \ dy/dt = 1 - x - \cos y$
4. $dx/dt = x + y^2,\ \ dy/dt = x + y$

In each of Problem 5 through 10:

(a) Determine all critical points of the given system of equations.
(b) Find the corresponding linear system near each critical point.
(c) Find the eigenvalues of each linear system. What conclusion can you then draw about the nonlinear system?
(d) *Draw a phase portrait of the nonlinear system to confirm your conclusions, or to extend them in those cases where the linear system does not provide definite information about the nonlinear system.

5. $dx/dt = (2 + x)(y - x),\ \ dy/dt = (4 - x)(y + x)$
6. $dx/dt = x - x^2 - xy,\ \ dy/dt = 3y - xy - 2y^2$
7. $dx/dt = 1 - y,\ \ dy/dt = x^2 - y^2$
8. $dx/dt = (2 + y)(y - 0.5x),\ \ dy/dt = (2 - x)(y + 0.5x)$
9. $dx/dt = x - x^2 - xy,\ \ dy/dt = y/2 - y^2/4 - 3xy/4$
10. $dx/dt = x + x^2 + y^2,\ \ dy/dt = y - xy$

11. Consider the autonomous system
$$dx/dt = y, \qquad dy/dt = x + 2x^3.$$
   (a) Show that the critical point (0,0) is a saddle point.
   (b) Sketch the trajectories for the corresponding linear system by integrating the equation for dy/dx. Show from the parametric form of the solution that the only trajectory on which $x \to 0, y \to 0$ as $t \to \infty$ is $y = -x$.
   (c) Determine the trajectories for the nonlinear system by integrating the equation for dy/dx. Sketch the trajectories for the nonlinear system that corresponds to $y = -x$ and $y = x$ for the linear system.

12. Consider the autonomous system
$$dx/dt = x, \qquad dy/dt = -2y + x^3.$$
   (a) Show that the critical point (0,0) is a saddle point.
   (b) Sketch the trajectories for the corresponding linear system, and show that the trajectory for which $x \to 0, y \to 0$ as $t \to \infty$ is given by $x = 0$.
   (c) Determine the trajectories for the nonlinear system for $x \neq 0$ by integrating the equation for dy/dx. Show that the trajectory corresponding to $x = 0$ for the linear system is unaltered, but that the one corresponding to $y = 0$ is $y = x^3/5$. Sketch several of the trajectories for the nonlinear system.

4.2 Liapunov's second method

In each of Problems 1 through 4 construct a suitable Liapunov function of the form $ax^2 + cy^2$, where a and c are to be determined. Then show that the critical point at the origin is of the indicated type.

1. $dx/dt = -x^3 + xy^2,\ \ dy/dt = -2x^2y - y^3;$ asymptotically stable
2. $dx/dt = x^3 - y^3,\ \ dy/dt = 2xy^2 + 4x^2y + 2y^3;$ unsatble
3. $dx/dt = -x^3 + 2y^3,\ \ dy/dt = -2xy^2;$ stable (at least)
4. $dx/dt = -x^3/2 + 2xy^2,\ \ dy/dt = -y^3;$ asymptotically stable

5. A generalization of the undamped pendulum equation is
$$d^2u/dt^2 + g(u) = 0,$$
where $g(0) = 0, g(u) > 0$ for $0 < u < k$, and $g(u) < 0$ for $-k < u < 0$; that is, $ug(u) > 0$ for $u \neq 0, -k < u < k$. Notice that $g(u) = \sin u$ has this property on $(-\pi/2, \pi/2)$.
   (a) Letting $x = u, y = du/dt$, write the equation as a system of two equations, and show that $x = 0, y = 0$ is a critical point.
   (b) Show that
$$V(x,y) = y^2/2 + \int_0^x g(s)ds,\ \ -k < x < k$$
is positive definite, and use this result to show that the critical point (0,0) is stable. Note that the Liapunov function V given by the equation corresponds to the energy function $V(x,y) = 1/2y^2 + (1 - \cos x)$ for the case $g(u) = \sin u$.

6. The Liénard equation is
$$d^2u/dt^2 + c(u)du/dt + g(u) = 0,$$
where g satisfies the conditions of Problem 5 and $c(u) \geq 0$. Show that the point $u = 0, du/dt = 0$ is a stable critical point.

7. A special case of the Liénard equation of Problem 6 is
$$d^2u/dt^2 + du/dt + g(u) = 0,$$
where g satisfies the conditions of Problem 5. Letting $x = u, y = du/dt$, show that the origin is a critical point of the resulting system. This equation can be interpreted as describing the motion of a spring–mass system with damping proportional to the velocity and a nonlinear restoring force. Using the Liapunov function of Problem 5, show that the origin is a stable critical point, but note that even with damping we cannot conclude asymptotic stability using this Liapunov function.

   Asymptotic stability of the critical point (0,0) can be shown by constructing a better Liapunov function. However, the analysis for a general function g is somewhat sophisticated and we only mention that an appropriate form for V is
$$V(x,y) = y^2/2 + Ayg(x) + \int_0^x g(s)ds,$$
where A is a positive constant to be chosen so that V is positive definite and $\dot{V}$ is negative definite. For the pendulum problem $g(x) = \sin x$; use V as given by the preceding equation with $A = 1/2$ to show that the origin is asymptotically stable.

Hint: Use $\sin x = x - \alpha x^3/3!$ and $\cos x = 1 - \beta x^2/2!$ where α and β depend on x, but $0 < \alpha < 1$ and $0 < \beta < 1$ for $-\pi/2 < x < \pi/2$; let $x = r\cos\theta$, $y = r\sin\theta$, and show that $\dot{V}(r\cos\theta, r\sin\theta) = -1/2r^2[1 + 1/2\sin 2\theta + h(r,\theta)]$, where $|h(r,\theta)| < 1/2$ if r is sufficiently small. To show that V is positive definite use $\cos x = 1 - x^2/2 + \gamma x^4/4!$, where γ depends on x, but $0 < \gamma < 1$ for $-\pi/2 < x < \pi/2$.

4.3 Periodic solutions and limit cycles

In each of Problems 1 through 6 an autonomous system is expressed in polar coordinates. Determine all periodic solutions, all limit cycles, and determine their stability characteristics.

1. $dr/dt = r^2(1 - r^2), \;\; d\theta/dt = 1$
2. $dr/dt = r(1 - r)^2, \;\; d\theta/dt = -1$
3. $dr/dt = \sin \pi r, \;\; d\theta/dt = 1$
4. $dr/dt = r(1 - r)(r - 2), \;\; d\theta/dt = -1$
5. $dr/dt = r(r - 1)(r - 3), \;\; d\theta/dt = 1$
6. $dr/dt = r|r - 1|(r - 3), \;\; d\theta/dt = -1$

7. If $x = r\cos\theta, y = r\sin\theta$, show that $y(dx/dt) - x(dy/dt) = -r^2(d\theta/dt)$.

8. Show that the system
$$dx/dt = -y + xf(r)/r, \quad dy/dt = x + yf(r)/r$$
has periodic solutions corresponding to the zeros of $f(r)$. What is the direction of motion on the closed trajectories in the phase plane?
Let $f(r) = r(r - 2)^2(r^2 - 4r + 3)$. Determine all periodic solutions and determine their stability characteristics.

9. Determine the periodic solutions, if any, of the system
$$\frac{dx}{dt} = y + \frac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2), \quad \frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2).$$

10. Show that the linear autonomous system
$$dx/dt = a_{11}x + a_{12}y, \quad dy/dt = a_{21}x + a_{22}y$$
does not have a periodic solution (other than $x = 0, y = 0$) if $a_{11} + a_{22} \neq 0$.

In each of Problems 11 and 12 show that the given system has no periodic solutions other than constant solutions.

11. $dx/dt = x + y + x^3 - y^2, \;\; dy/dt = -x + 2y + x^2y + y^3/3$
12. $dx/dt = -3x - 3y - xy^2, \;\; dy/dt = 2y + x^3 - x^2y$

13. Consider the system of equations

$$x' = \mu x + y - x(x^2 + y^2), \quad y' = -x + \mu y - y(x^2 + y^2),$$

where μ is a parameter.

(a) Show that the origin is the only critical point.

(b) Find the linear system that approximates the equation near the origin and find its eigenvalues. Determine the type and stability of the critical point at the origin. How does this classification depend on μ?

(c) Rewrite the equation in polar coordinates.

(d) Show that when $\mu > 0$, there is a periodic solution $r = \sqrt{\mu}$. By solving the system found in part (c), or by plotting numerically computed approximate solutions, conclude that this periodic solution attracts all other nonzero solutions.

Note: As the parameter μ increases through the value zero, the previously asymptotically stable critical point at the origin loses its stability, and simultaneously a new asymptotically stable solution (the limit cycle) emerges. Thus the point $\mu = 0$ is a bifurcation point; this type of bifurcation is called **Hopf bifurcation**.

5.1 Series solutions near ordinary point I

In each of Problems 1 through 6:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

1. $y'' - y = 0,\ \ x_0 = 0$
2. $y'' - xy' - y = 0,\ \ x_0 = 0$
3. $y'' - xy' - y = 0,\ \ x_0 = 1$
4. $y'' + k^2x^2y = 0,\ \ x_0 = 0,\ \ k\ a\ constant$
5. $(1 - x)y'' + y = 0,\ \ x_0 = 0$
6. $(2 + x^2)y'' - xy' + 4y = 0,\ \ x_0 = 0$

7. Show directly, using the ratio test, that the two series solutions of Airy's equation about $x = 0$ converges for all x.

8. **The Hermite Equation.** The equation
$$y'' - 2xy' + \lambda y = 0, \qquad -\infty < x < \infty,$$
where λ is a constant, is known as the Hermite equation. It is an important equation in mathematical physics.
   (a) Find the first four terms in each of two solutions about $x = 0$ and show that they form a fundamental set of solutions.
   (b) Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solution for $\lambda = 0, 2, 4, 6, 8$ and 10. Note that each polynomial is determined only up to a multiplicative constant.
   (c) The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), \cdots, H_5(x)$.

9. Consider the initial value problem $y' = \sqrt{1 - y^2}, y(0) = 0$.
   (a) Show that $y = \sin x$ is the solution of this initial value problem.
   (b) Look for a solution of the initial value problem in the form of a power series about $x = 0$. Find the coefficients up to the term in x^3 in this series.

5.2 Series solutions near ordinary point II

In each of Problem 1 and 2 determine the $\phi''(x_0)$, $\phi'''(x_0)$ and $\phi^{(4)}(x_0)$ for the given point x_0 if $y = \phi(x)$ is a solution of the given initial value problem.

1. $y'' + xy' + y = 0;\ \ y(0) = 1,\ y'(0) = 0$
2. $y'' + x^2y' + (\sin x)y = 0;\ \ y(0) = a_0,\ y'(0) = a_1$

In each of Problem 3 and 4 determine the lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation.

3 $y'' + 4y' + 6xy = 0; \quad x_0 = 0, \quad x_0 = 4$

4 $(x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad x_0 = 4, \quad x_0 = -4, \ x_0 = 0$

5. Determine a lower bound for the radius of convergence of series solutions about the given x_0 for each of the differential equations in Problems 1 through 6 of **Section 5.1.**

6. **The Chebyshev Equation.** The Chebyshev differential equation is
$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$
where α is a constant.
   (a) Determine two solutions in powers of x for $|x| < 1$ and show that they form a fundamental set of solutions.
   (b) Show that if α is nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called the Chebyshev polynomials. They are very useful in problems that require a polynomial approximation to a function defined on $-1 \le x \le 1$.
   (c) Find polynomial solution for each of the cases $\alpha = n = 0, 1, 2, 3$.

The Legendre Equation. Problems 7 through 10 deal with the Legendre equation
$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$
The point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \le -1$, then the substitution $\alpha = -(1 + \gamma)$, where $\gamma \ge 0$, leads to the Legendre equation $(1 - x^2)y'' - 2xy' + \gamma(\gamma + 1)y = 0$.

7. Show that two solutions of the Legendre equation for $|x| < 1$ are
$$y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 + \sum_{m=3}^{\infty}(-1)^m \frac{\alpha \cdots (\alpha - 2m + 2)(\alpha + 1)\cdots(\alpha + 2m - 1)}{(2m)!}x^{2m},$$
$$y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 + \sum_{m=3}^{\infty}(-1)^m \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}.$$

8. Show that, if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0, 2,$ and 4. Show that, if α is a positive odd integer $2n + 1$, the series solution y_2 reduces to a polynomial of degree $2n + 1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1, 3,$ and 5.

9. It can be shown that the general formula for $P_n(x)$ is
$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!\,(n-k)!\,(n-2k)!}x^{n-2k},$$

where $\lfloor n/2 \rfloor$ denotes the greatest integer less than or equal to $n/2$. By observing the form of $P_n(x)$ for n even and n odd, show that $P_n(-1) = (-1)^n$.

10. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace's equation (the potential equation) in spherical coordinates, we encounter the equation
$$\frac{d^2F(\varphi)}{d\varphi^2} + \cot\varphi \frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0, \qquad 0 < \varphi < \pi,$$
where n is positive integer. Show that the change of variables $x = \cos\varphi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\arccos x)$.

5.3 Euler equations; Regular singular points

In each of Problem 1 and 2 determine the general solution of the given differential equation that is valid in any interval not including singular point.

1. $x^2y'' + 4xy' + 2y = 0$
2. $(x+1)^2y'' + 3(x+1)y' + 0.75y = 0$

In each of Problem 1 and 2 find all singular points of the given equation and determine whether each one is regular or irregular.

3. $xy'' + (1-x)y' + xy = 0$
4. $x^2(1-x)^2y'' + 2xy' + 4y = 0$

5. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + (5/2)y = 0$ approaches zero as $x \to 0$.

6. Find γ so that the solution of the initial value problem $x^2y'' - 2y = 0, y(1) = 1, y'(1) = \gamma$ is bounded as $x \to 0$.

7. Consider the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$. Find conditions on α and β so that:
   (a) All solutions approach zero as $x \to 0$.
   (b) All solutions are bounded as $x \to 0$.
   (c) All solutions approach zero as $x \to \infty$.
   (d) All solutions are bounded as $x \to \infty$.
   (e) All solutions are bounded both as $x \to 0$ and as $x \to \infty$.

8. Using the method of reduction of order, show that if r_1 is a repeated root of
$$r(r-1) + \alpha r + \beta = 0,$$
then x^{r_1} and $x^{r_1}\ln x$ are solutions of $x^2y'' + \alpha xy' + \beta y = 0$ for $x > 0$.

5.4 Series solutions near singular point I

In each of Problem 1 through 4:

(a) Show that the given differential equation has a regular singular point at $x = 0$.
(b) Determine the indication equation, the recurrence relation, and the roots of the indicial equation.
(c) Find the series solution ($x > 0$) corresponding to the larger root.
(d) If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

1. $2xy'' + y' + xy = 0$
2. $x^2y'' + xy' + (x^2 - 1/9)y = 0$
3. $3x^2y'' + 2xy' + x^2y = 0$
4. $xy'' + y' - y = 0$

5. The Legendre equation of order α is
$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$
The solution of this equation near the ordinary point $x = 0$ was discussed in Problems 7 and 10 of Section 5.2. It can be shown that $x = \pm 1$ are regular singular points.
(a) Determine the indicial equation and its roots for the point $x = 1$.
(b) Find a series solution in powers of $x - 1$ for $x - 1 > 0$.
Hint: Write $1 + x = 2 + (x - 1)$ and $x = 1 + (x - 1)$. Alternatively, make the change of variable $x - 1 = t$ and determine a series solution in powers of t.

6. The Chebyshev equation is
$$(1 - x^2)y'' - xy' + \alpha^2 y = 0.$$
where α is constant.
(a) Show that $x = 1$ and $x = -1$ are regular singular points, and find the exponents at each of these singularities.
(b) Find two solutions about $x = 1$.

7. The Laguerre differential equation is
$$xy'' + (1 - x)y' + \lambda y = 0.$$
(a) Show that $x = 0$ is a regular singular point.
(b) Determine the indicial equation, its roots, and the recurrence relation.
(c) Find one solution ($x > 0$). Show that if $\lambda = m$, a positive integer, this solution reduces to a polynomial. When properly normalized, this polynomial is known as the Laguerre polynomial, $L_m(x)$.

8. The Bessel equation of order zero is
$$x^2y'' + xy' + x^2y = 0.$$
(a) Show that $x = 0$ is a regular singular point.
(b) Show that the roots of the indicial equation are $r_1 = r_1 = 0$.
(c) Show that one solution for $x > 0$ is
$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$
(d) Show that the series for $J_0(x)$ converges for all x. The function J_0 is known as the Bessel function of the first kind of order zero.

5.5 Series solutions near singular point II

In each of Problem 1 through 6:

(a) Find all the regular singular points of the given differential equation.
(b) Determine the indicial equation and the exponents at the singularity for each regular singular point.

1. $xy'' + 2xy' + 6e^x y = 0$
2. $x^2 y'' - x(2+x)y' + (2+x^2)y = 0$
3. $x(x-1)y'' + 6x^2 y' + 3y = 0$
4. $y'' + 4xy' + 6y = 0$
5. $2x(x+2)y'' + y' - xy = 0$
6. $x^2 y'' + 3\,(\sin x)y' - 2y = 0$

In each of Problem 7 through 10:

(a) Show that $x = 0$ is a regular singular point of the given differential equation.
(b) Find the exponents at the singular point $x = 0$.
(c) Find the first three nonzero terms in each of two solutions (not multiples of each others) about $x = 0$.

7. $xy'' + y' - y = 0$
8. $xy'' + 2xy' + 6e^x y = 0$
9. $xy'' + y = 0$
10. $x(x-1)y'' + 6x^2 y' + 3y = 0$

5.6 Bessel equation

1. Consider the Bessel equation of order ν
$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad x > 0,$$
where ν is real and positive.
   (a) Show that $x = 0$ is a regular singular point and that the roots of the indicial equation are ν and $-\nu$.
   (b) Corresponding to the larger root ν, show that one solution is
$$y_1(x) = x^\nu \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(1+\nu)\cdots(n+\nu)} \left(\frac{x}{2}\right)^{2n}\right].$$
   (c) If 2ν is not an integer, show that a second solution is
$$y_2(x) = x^{-\nu} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(1-\nu)\cdots(n-\nu)} \left(\frac{x}{2}\right)^{2n}\right].$$
   Note that $y_1(x) \to 0$ as $x \to 0$, and that $y_2(x)$ is unbounded as $x \to 0$.
   (d) Verify by direct methods that the power series in the expressions for $y_1(x)$ and $y_2(x)$ converge absolutely for all x. Also verify that y_2 is a solution provided only that ν is not an integer.

2. Find two solutions (not multiples of each other) of Bessel equation of order zero.
$$x^2 y'' + xy' + x^2 y = 0, \quad x > 0.$$

3. Find two solutions (not multiples of each other) of Bessel equation of order one-half.
$$x^2 y'' + xy' + (x^2 - 1/4)y = 0, \quad x > 0.$$

4. Find two solutions (not multiples of each other) of Bessel equation of order one.
$$x^2y'' + xy' + (x^2 - 1)y = 0, \quad x > 0.$$

5. Show that the Bessel equation of order one-half (Reduction of order)
$$x^2y'' + xy' + (x^2 - 1/4)y = 0, \quad x > 0.$$
can be reduced to the equation
$$v'' + v = 0$$
by the change of dependent variable $y = x^{-1/2}v(x)$. From this conclude that $y_1 = x^{-1/2}\cos x$ and $y_2 = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.