Introduction to mechanical engineering lecture notes

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Contents

1 A short summary of the basics

1.1 Physical quantities, units and working with units

The value of a **physical quantity** Q is expressed as the product of a numerical value Q and a unit of measurement $[Q]$:

$$Q = Q \times [Q] \tag{1}$$

For example, if the temperature T of a body is quantified (measured) as 25 degrees Celsius this is written as:

$$T = 25 \times^o C = 25^o C, \tag{2}$$

where T is the symbol of the physical quantity "temperature", 25 is the numerical factor and ^oC is the unit.

By convention, physical quantities are organized in a dimensional system built upon base quantities, each of which is regarded as having its own dimension. The seven base quantities of the International System of Quantities (ISQ) and their corresponding SI units are listed in Table 1. Other conventions may have a different number of fundamental units (e.g. the CGS and MKS systems of units).

Name	Symbol for quantity	Symbol for dimension	SI base unit	Symbol for unit
Length	l, x, r, etc.	L	meter	m
Time	t	T	second	s
Mass	m	M	kilogram	kg
Electric current	I, i	I	ampere	A
Thermodynamic temperature	T	θ	kelvin	K
Amount of substance	n	N	mole	mol
Luminous intensity	I_v	J	candela	cd

Table 1: International System of Units base quantities

All other quantities are derived quantities since their dimensions are derived from those of base quantities by multiplication and division. For example, the physical quantity velocity is derived from base quantities length and time and has dimension L/T. Some derived physical quantities have dimension 1 and are said to be *dimensionless quantities*.

The International System of Units (SI) specifies a set of unit *prefix*es known as SI prefixes or metric prefixes. An SI prefix is a name that precedes a basic unit of measure to indicate a decimal multiple or fraction of the unit. Each prefix has a unique symbol that is prepended to the unit symbol, see Table 2.

Prefix	Symbol	10^n
giga	G	10^9
mega	M	10^6
kilo	k	10^3
hecto	h	10^2
deca	da	10^1
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}

Table 2: International System of Units prefixes.

A quantity is called:

extensive when its magnitude is additive for subsystems (volume, mass, etc.)

intensive when the magnitude is independent of the extent of the system (temperature, pressure, etc.)

Units can be used as numbers in the sense that you can add, subtract, multiply and divide them - with care. Much confusion can be avoided if you work with units as though they were symbols in algebra. For example:

- Multiply units along with numbers:

  (5 m) × (2 sec) = (5 × 2) × (m × sec) = 10 m sec.

  The units in this example are meters times seconds, pronounced as 'meter seconds' and written as 'm sec'.

- Divide units along with numbers:

  (10 m) / (5 sec) = (10 / 5) × (m / sec) = 2 m/sec.

  The units in this example are meters divided by seconds, pronounced as 'meters per second' and written as 'm/sec'. This is a unit of speed.

- Cancel when you have the same units on top and bottom:

  (15 m) / (5 m) = (15 / 5) × (m / m) = 3.

In this example the units (meters) have cancelled out, and the result has no units of any kind! This is what we call a ‘pure’ number. It would be the same regardless what system of units were used.

- When adding or subtracting, convert both numbers to the same units before doing the arithmetic:

  (5 m) + (2 cm) = (5 m) + (0.02 m) = (5 + 0.02) m = 5.02 m.

  Recall that a ‘cm’, or centimeter, is one hundredth of a meter. So 2 cm = (2 / 100) m = 0.02 m.

- You can’t add or subtract two numbers unless you can convert them both to the same units:

  (5 m) + (2 sec) = ???

1.2 Understanding the words "steady-state" and "unsteady"

TODO

1.3 Linear motion

Linear motion is motion along a straight line, and can therefore be described mathematically using only one spatial dimension. It can be uniform, that is, with constant velocity (zero acceleration), or non-uniform, that is, with a variable velocity (non-zero acceleration). The motion of a particle (a point-like object) along the line can be described by its position x, which varies with t (time).

An example of linear motion is that of a ball thrown straight up and falling back straight down.

The average velocity v during a finite time span of a particle undergoing linear motion is equal to $\overline{v} = \sum x / \sum t$, where $\sum x$ is the total displacement and $\sum t$ denotes the time needed.

The instantaneous velocity of a particle in linear motion may be found by differentiating the position x with respect to the time variable t: $v = dx/dt$. The acceleration may be found by differentiating the velocity: $a = dv/dt$. By the fundamental theorem of calculus the converse is also true: to find the velocity when given the acceleration, simply integrate the acceleration with respect to time; to find displacement, simply integrate the velocity with respect to time.

This can be demonstrated graphically. The gradient of a line on the displacement time graph represents the velocity. The gradient of the velocity time graph gives the acceleration while the area under the velocity time graph gives the displacement. The area under an acceleration time graph gives the velocity.

1.4 Circular motion

Circular motion is rotation along a circle: a circular path or a circular orbit. It can be uniform, that is, with constant angular rate of rotation, or non-uniform, that is, with a changing rate of rotation.

Examples of circular motion are: an artificial satellite orbiting the Earth in geosynchronous orbit, a stone which is tied to a rope and is being swung in circles (cf. hammer throw), a racecar turning through a curve in a race track, an electron moving perpendicular to a uniform magnetic field, a gear turning inside a mechanism. *Circular motion is accelerated even if the angular rate of rotation is constant, because the object's velocity vector is constantly changing direction.* Such change in direction of velocity involves acceleration of the moving object by a centripetal force, which pulls the moving object towards the center of the circular orbit. Without this acceleration, the object would move in a straight line, according to Newton's laws of motion.

For motion in a circle of radius R, the circumference of the circle is $C = 2\pi R$. If the period for one rotation is T, the angular rate of rotation, also known as *angular velocity*, $\omega\,[rad/s]$ is:

$$\omega = \frac{2\pi}{T}. \tag{3}$$

In mechanical engineering, the *revolution number* is often used:

$$n = \frac{\omega}{2\pi} \times 60 \left[rpm = \frac{\text{number of rotations}}{\text{minute}} \right] \tag{4}$$

The speed of the object travelling the circle is

$$v = \frac{2\pi R}{T} = R\omega. \tag{5}$$

The angle θ swept out in a time t is

$$\theta = 2\pi \frac{t}{T} = \omega t. \tag{6}$$

The acceleration due to change in the direction of the velocity is found by analysing the change of the velocity vector in (small) time interval Δt. As $\omega = \text{const.}$, we have $|v_1| = |v_2| := v$. From the triangle we see that

$$\frac{\frac{\Delta v}{2}}{v} = \sin\frac{\Delta\varphi}{2} \approx \frac{\Delta\varphi}{2} \quad \text{for} \quad \varphi < 5^o \quad \to \quad \frac{\Delta v}{v} = \Delta\varphi = \omega\Delta t. \tag{7}$$

Thus, we have

$$a = \frac{\Delta v}{\Delta t} = \omega v = R\omega^2 = \frac{v^2}{R} \tag{8}$$

and is directed radially inward.

The *angular acceleration* $\varepsilon\,[rad/s^2]$ is

$$\varepsilon = \frac{\Delta\omega}{\Delta t}. \tag{9}$$

1.5 Newton's first law

Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.

This law states that if the resultant force (the vector sum of all forces acting on an object) is zero, then the velocity of the object is constant. Consequently:

- An object that is at rest will stay at rest unless an unbalanced force acts upon it.
- An object that is in motion will not change its velocity unless an unbalanced force acts upon it.

Newton placed the first law of motion to establish *frames of reference* for which the other laws are applicable. The first law of motion postulates the existence of at least one frame of reference called a Newtonian or inertial reference frame, relative to which the motion of a particle not subject to forces is a straight line at a constant speed.

1.6 Newton's second law

The second law states that the net force on a particle is equal to the time rate of change of its linear momentum p in an inertial reference frame:

$$F = \frac{dp}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma, \tag{10}$$

where we assumed constant mass. Thus, the net force applied to a body produces a proportional acceleration.

For circular motion, we have

$$M = \theta\varepsilon, \tag{11}$$

with $M\,[Nm]$ being the torque $M = Fr$, ε denotes angular acceleration and $\theta\,[kgm^2]$ is the *moment of inertia.*

1.7 The moment of inertia

The moment of inertia of an object about a given axis describes how difficult it is to change its angular motion about that axis. The standard notation is θ with the actual axis in the subscript, e.g. θ_x meaning moment of inertia with respect to axis x.

Consider a point of mass m rotating around an axis with circumferential velocity v. The angular velocity is ω and the radius of the circle is r. The kinetic energy is

$$E = \frac{1}{2}mv^2 = \frac{1}{2}m\,(r\omega)^2 = \frac{1}{2}\omega^2 \underbrace{mr^2}_{\theta}. \tag{12}$$

Some equations for the moment of inertia:

- a **mass point** rotating on a circle of radius r: $\theta = mr^2$
- a thin **ring** of radius r rotating around its own axis: $\theta = mr^2$
- a thin **disc** of radius r rotating around its own axis: $\theta = \frac{1}{2}mr^2$
- a thin rod of mass m and length l, rotating around the axis which passes through its **center** and is perpendicular to the rod: $\theta = \frac{1}{12}ml^2$
- a thin rod of mass m and length l, rotating around the axis which passes through its **end** and is perpendicular to the rod: $\theta = \frac{1}{3}ml^2$
- a **solid ball** of mass m and radius r, rotating around an axis which passes through the center: $\theta = \frac{2}{5}mr^2$

Let us calculate the moment of inertia of a disc of height b, radius R and uniform density ρ at its own axis. We divide the radius into $N+1$ rings: $r_i = iR/N = i\Delta r$. The moment of inertia of the i^{th} ring is

$$\theta_i = m_i r_i^2 = \underbrace{\underbrace{2r_i\pi}_{\text{circumference}} \quad \Delta r}_{\text{area}}\; b\rho\, r_i^2 = 2i\frac{R}{N}\pi\frac{R}{N}\,b\rho\,\left(i\frac{R}{N}\right)^2 \tag{13}$$

By summing up these rings we obtain

$$\begin{aligned}\theta_{\text{disc}} &= \lim_{N\to\infty}\sum_{i=1}^{N}\theta_i = \lim_{N\to\infty} 2\left(\frac{R}{N}\right)^4 \pi b\rho\sum_{i=1}^{N} i^3 \\ &= \lim_{N\to\infty} 2\left(\frac{R}{N}\right)^4 \pi b\rho\frac{1}{4}N^2(1+N^2) = 2R^4\pi b\rho\frac{1}{4} = \frac{1}{2}mR^2 \end{aligned} \tag{14}$$

The *parallel axis theorem* (or Huygens-Steiner theorem) can be used to determine the moment of inertia of a rigid body about *any axis*, given the moment of inertia of the object about the parallel axis through the object's centre of mass and the perpendicular distance (r) between the axes. The moment of inertia about the new axis z is given by:

$$\theta_z = \theta_{cm} + mr^2 \tag{15}$$

where θ_{cm} is the moment of inertia of the object about an axis passing through its centre of mass, m is the object's mass and r is the perpendicular distance between the two axes. For example, let us compute the moment of inertia of a thin rod rotating around the axis which passes through its **end**:

$$\theta_{end} = \theta_{cm} + m\left(\frac{l}{2}\right)^2 = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 = \frac{1}{3}ml^2. \tag{16}$$

Finally, the moment of inertia of an object can be computed simply as the sum of moments of inertia of its "building" objects.

1.8 Work

In physics, *mechanical work* is the amount of energy transferred by a force acting through a distance. In the simplest case, if the force and the displacement are parallel and constant, we have

$$W = Fs. \tag{17}$$

It is a scalar quantity, with SI units of *joules*. If the direction of the force and the displacement do not coincide (e.g. when pulling a bob up to a hill) - but they are still constant - one has to take the parallel components:

$$W = \mathbf{F} \cdot \mathbf{v} = |\mathbf{F}|\,|\mathbf{v}|\cos\theta = Fvcos\theta, \tag{18}$$

where θ is the angle between the force and the displacement vector and $\cdot$ stands for the dot product of vectors.

In situations where the force changes over time, or the path deviates from a straight line, equation (17) is not generally applicable although it is possible to divide the motion into small steps, such that the force and motion are well approximated as being constant for each step, and then to express the overall work as the sum over these steps. Mathematically, the calculation of the work needs the evaluation of the following line integral:

$$W_C = \int_C \mathbf{F} \cdot d\mathbf{s}, \tag{19}$$

where C is the path or curve traversed by the object; $\mathbf{F}$ is the force vector; and $\mathbf{s}$ is the position vector. Note that the result of the above integral depends on the path and only from the endpoints. This is typical for systems in which losses (e.g. friction) are present (similarly as the actual fare of a taxi from point A to B depends heavily on the route the driver chooses).

1.9 Energy

Energy is a quantity that is often understood as the ability to perform work. This quantity can be assigned to any particle, object, or system of objects as a consequence of its physical state.

Energy is a scalar physical quantity. In the International System of Units (SI), energy is measured in joules, but in some fields other units such as kilowatt-hours and kilocalories are also used. Different forms of energy include kinetic, potential, thermal, gravitational, sound, elastic and electromagnetic energy.

Any form of energy can be transformed into another form. When energy is in a form other than thermal energy, it may be transformed with good or even perfect efficiency, to any other type of energy, however, during this conversion a portion of energy is usually lost because of losses such as friction, imperfect heat isolation, etc.

In mechanical engineering, we are mostly concerned with the following types of energy:

- potential energy: $E_p = mgh$
- kinetic energy: $E_k = \frac{1}{2}mv^2$
- internal energy: $E_t = c_p mT$ (with a huge number of simplifications...)

Although the total energy of an *isolated* system does not change with time[1], its value may depend on the frame of reference. For example, a seated passenger in a moving airplane has zero kinetic energy relative to the airplane, but non-zero kinetic energy (and higher total energy) relative to the Earth.

A *closed* system interacts with its surrounding with mechanical work (W) and heat transfer (Q). Due to this interaction, the energy of the system changes:

$$\Delta E = W + Q, \tag{20}$$

where work is positive if the system's energy increases (e.g. by lifting objects their potential energy increases) and heat transfer is positive if the temperature of the system increases.

[1] There is a fact, or if you wish, a law, governing all natural phenomena that are known to date. There is no known exception to this lawit is exact so far as we know. The law is called the conservation of energy. It states that there is a certain quantity, which we call energy, that does not change in manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same. The Feynman Lectures on Physics

1.10 Power

Power is the rate at which work is performed or energy is converted. If ΔW is the amount of work performed during a period of time of duration Δt, the average power $\overline{P}$ over that period is given by

$$\overline{P} = \frac{\Delta W}{\Delta t}. \quad (21)$$

The average power is often simply called "power" when the context makes it clear.

The instantaneous power is then the limiting value of the average power as the time interval Δt approaches zero. In the case of constant power P, the amount of work performed during a period of duration T is $W = PT$. Depending on the actual machine, we have

mechanical (linear motion) power: $P = Fv \left[N\frac{m}{s}\right]$

mechanical (circular motion) power: $P = M\omega \left[Nm\frac{rad}{s}\right]$

electrical power: $P = UI \ [VA]$

hydraulic power: $P = Q\Delta p \left[\frac{m^3}{s}Pa\right]$

The dimension of power is energy divided by time J/s. The SI unit of power is the watt (W), which is equal to one joule per second. A common non-SI unit of power is horsepower (hp), $1hp = 0.73549875kW$.

1.11 Problems

Problem 1.1 A spring with stiffness $s = 100\,N/mm$ is compressed from its initial length of $L_0 = 20cm$ to $L_1 = 10cm$.

- Calculate the force. ($F = 10kN$)
- Calculate the work. ($W = 0.5kJ$)

Problem 1.2 We drive by car for 4 hours, after which we refuel $32l$ of gasoline. The car has a 55kW motor (75hp) and it can be assumed that during the journey this was the useful power. The heating value of gasoline is $35\,MJ/l$.

- Calculate the useful work ($W_u = 220kWh = 792MJ$), input energy ($E_i = 1120MJ$) and efficiency ($\eta = 70.7\%$).

Problem 1.3 A $210MW$ coal plant consumes $4100t$ of coal per day. The heating value of lignite is $17MJ/kg$.

- Calculate the efficiency of the plant ($\eta = 26\%$).

Worked problem 1.4 A rotating wheel of a vehicle is stopped by two brakes as seen in Figure 1. The friction coefficient is $\mu = 0.13$, the diameter of the wheel is $910mm$, the initial velocity of the vehicle was $65km/h$. The pushing force is $F = 6000N$.

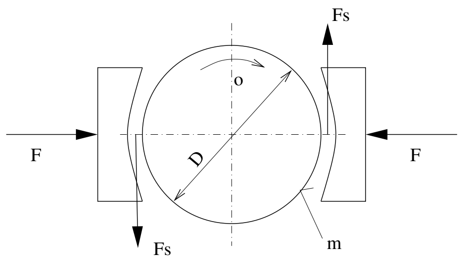

Figure 1: Braking a rotating wheel.

- Calculate the friction force.
  $F_f = \mu F = 0.78kN$.

- Calculate the (overall) braking torque acting on the wheel.
  $M_f = 2 \times F_f \frac{D}{2} = 0.7098kNm$

- Calculate the power of braking at the start of the breaking.
  $P = M_f \omega_0 = M_f \frac{2v_0}{D} = 28.17kW$

- Assuming constant torque and linearly decreasing velocity (i.e. constant deceleration), compute the time needed to stop a 20t vehicle with six braked weels.
  The initial kinetic energy of the vehicle is $E_k = \frac{1}{2}mv_0^2 = 3.26MJ$.
  The overall work done by the six brakes $W = 6 \times M_f T \frac{\omega}{2}$ (T is yet unknown).
  The initial kinetic energy is fully dissipated by the braking work: $E_k = W \rightarrow T = 38.6s$

2 Steady-state operation of machines

2.1 The sliding friction force due to dry friction

Dry friction resists relative lateral motion of two solid surfaces in contact. The two regimes of dry friction are static friction between non-moving surfaces, and kinetic friction (sometimes called sliding friction or dynamic friction) between moving surfaces. Coulomb friction is an approximate model used to calculate the force of dry friction:

$$|F_f| \leq \mu N. \tag{22}$$

where

- F_f is the force exerted by friction (in the case of equality, the maximum possible magnitude of this force).
- μ is the coefficient of friction, which is an empirical property of the contacting materials,
- N is the normal force exerted between the surfaces.

The Coulomb friction may take any value from zero up to μN, and the direction of the frictional force against a surface is opposite to the motion that surface would experience in the absence of friction. Thus, in the static case, the frictional force is exactly what it must be in order to prevent motion between the surfaces; it balances the net force tending to cause such motion. In this case, rather than providing an estimate of the actual frictional force, the Coulomb approximation provides a threshold value for this force, above which motion would commence. This maximum force is known as traction.

The force of friction is always exerted in a direction that opposes movement (for kinetic friction) or potential movement (for static friction) between the two surfaces. For example, a curling stone sliding along the ice experiences a kinetic force slowing it down. For an example of potential movement, the drive wheels of an accelerating car experience a frictional force pointing forward; if they did not, the wheels would spin, and the rubber would slide backwards along the pavement. Note that it is not the direction of movement of the vehicle they oppose, it is the direction of (potential) sliding between tire and road. In the case of kinetic friction, the direction of the friction force may or may not match the direction of motion: a block sliding atop a table with rectilinear motion is subject to friction directed along the line of motion; an automobile making a turn is subject to friction acting perpendicular to the line of motion (in which case it is said to be 'normal' to it). The direction of the static friction force can be visualized as directly opposed to the force that would otherwise cause motion, were it not for the

Materials		Dry and clean	Lubricated
Aluminum	Steel	0.61	
Copper	Steel	0.53	
Brass	Steel	0.51	
Cast iron	Copper	1.05	
Cast iron	Zinc	0.85	
Concrete (wet)	Rubber	0.30	
Concrete (dry)	Rubber	1.0	
Concrete	Wood	0.62	
Copper	Glass	0.68	
Glass	Glass	0.94	
Metal	Wood	0.2-0.6	0.2
Polythene	Steel	0.2	0.2
Steel	Steel	0.80	0.16
Steel	Teflon	0.04	0.04
Teflon	Teflon	0.04	0.04
Wood	Wood	0.25-0.5	0.2

Table 3: Approximate coefficients of friction

static friction preventing motion. In this case, the friction force exactly cancels the applied force, so the net force given by the vector sum, equals zero. It is important to note that in all cases, Newton's first law of motion holds.

2.2 Rolling resistance

Rolling resistance, sometimes called rolling friction or rolling drag, is the resistance that occurs when a round object such as a ball or tire rolls on a flat surface, in steady velocity straight line motion. It is caused mainly by the deformation of the object, the deformation of the surface, or both. (Additional contributing factors include wheel radius, forward speed, surface adhesion, and relative micro-sliding between the surfaces of contact.) It depends very much on the material of the wheel or tire and the sort of ground.

For example, rubber will give a bigger rolling resistance than steel. Also, sand on the ground will give more rolling resistance than concrete. A moving wheeled vehicle will gradually slow down due to rolling resistance including that of the bearings, but a train car with steel wheels running on steel rails will roll farther than a bus of the same mass with rubber tires running on tarmac. The coefficient of rolling resistance is generally much smaller for tires or balls than the coefficient of sliding friction.

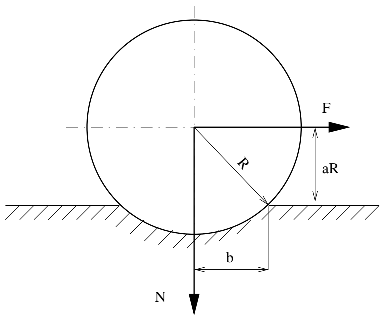

Figure 2: Hard wheel rolling on and deforming a soft surface.

The force of rolling resistance can also be calculated by:

$$F = \frac{Nb}{R} = C_{rr}N \tag{23}$$

where

- F is the rolling resistance force,
- R is the wheel radius,
- b is the rolling resistance coefficient or coefficient of rolling friction with dimension of length,
- $C_{rr} = b/R$ is the *c*oefficient of *r*olling *r*esistance (dimensionless number), and
- N is the normal force.

C_{rr}	b	Description
0.0002...0.0010	0.5 mm	Railroad steel wheel on steel rail
	0.1mm	Hardened steel ball bearings on steel
0.0025		Special Michelin solar car/eco-marathon tires
0.005		Tram rails standard dirty with straights and curves[citation needed]
0.0055		Typical BMX bicycle tires used for solar cars
0.0062...0.015		Car tire measurements
0.010...0.015		Ordinary car tires on concrete
0.3		Ordinary car tires on sand

Table 4: Approximate coefficients of rolling resistance

In usual cases, the normal force on a single tire will be the mass of the object that the tires are supporting divided by the number of wheels, plus the mass of the wheel, times the gravitational acceleration. In other words, the normal force is equal to the weight of the object being supported, if the wheel is on a horizontal surface.

2.3 Statics of objects on inclined planes (restoring forces)

To calculate the forces on an object placed on an inclined plane, consider the three forces acting on it:

- The normal force (N) exerted on the body by the plane due to the force of gravity,
- The force due to gravity (mg, acting vertically downwards) and
- the frictional force (F_f) acting parallel to the plane.

We can decompose the gravitational force into two vectors, one perpendicular to the plane and one parallel to the plane. Since there is no movement perpendicular to the plane, the component of the gravitational force in this direction ($mg \cos \alpha$) must be equal and opposite to normal force exerted by the plane, N plus the normal component of the force F:

$$mg \cos \alpha = F \sin \beta + N. \tag{24}$$

The remaining component of the gravitational force parallel to the surface ($mg \sin \alpha$) plus the friction force equals the the "pulling" force F_t:

$$mg \sin \alpha + F_f = F \cos \beta \tag{25}$$

By definition, the friction force is

$$F_f = \mu N. \tag{26}$$

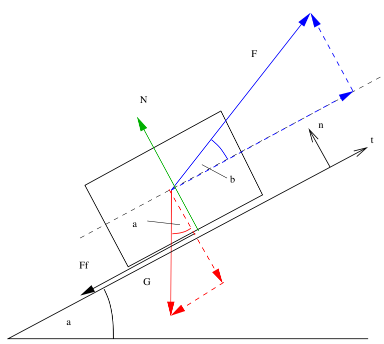

Figure 3: Object on an inclined plane.

Combining these three equations, we arrive at the following equation for the traction force F:

$$F = mg\frac{\sin\alpha + \mu\cos\alpha}{\cos\beta + \mu\sin\beta}. \tag{27}$$

Let us describe the three different cases while varying the force:

- If the actual traction force is smaller than F (but positive), the object is at rest. The value of the friction force is such that the body stays at rest: $F_f = mg\sin\alpha$ and points upwards to balance the tangential component of G.
- If the traction force is exactly F, the body is either at rest or moves with constant *arbitrary* velocity.
- Finally, for traction force values beyond F the object experiences a constant acceleration.

One might want to compute the β^* angle for which the smallest F force is needed to move the object:

$$\begin{aligned} 0 = \frac{dF}{d\beta} &= \underbrace{mg\left(\sin\alpha + \mu\cos\alpha\right)}_{\text{const.}}\frac{d}{d\beta}\left(\cos\beta + \mu\sin\beta\right)^{-1} \\ &= \text{const.}(-1)\frac{-\sin\beta^* + \mu\cos\beta^*}{(\cos\beta^* + \mu\sin\beta^*)^{-2}}. \quad \rightarrow \quad \beta^* = \arctan\mu \end{aligned} \tag{28}$$

Note that our equations are valid only if $N > 0$, i.e.

$$G\cos\alpha \geq F_{max}\sin\beta \tag{29}$$

In other words, the maximal β_{max} value before the force F lifts the object is (the condition is $F = F_{max}$):

$$\frac{\sin\alpha + \mu\cos\alpha}{\cos\beta_{max} + \mu\sin\beta_{max}} = \frac{\cos\alpha}{\sin\beta_{max}} \quad \rightarrow \quad \beta_{max} = \operatorname{arc}\cot\left(\tan\alpha\right) = \frac{\pi}{2} - \alpha. \tag{30}$$

Finally, let us define the efficiency of the traction. The useful work is the vertical displacement $\Delta h = L\sin\alpha$ while the input work is the work done by tangential component of the force, i.e. F_t:

$$\eta = \frac{\Delta E_{pot}}{W_{F_t}} = \frac{mg\Delta h}{F_t L} = \frac{mgL\sin\alpha}{mgL\sin\alpha + F_f L}, \tag{31}$$

whose maximum value occurs if $F_f = 0$, i.e. when the normal component vanishes. Hence the maximum-efficiency point corresponds to the β_{max} angle, i.e. when the normal component of force F balances the gravity and no friction occurs.

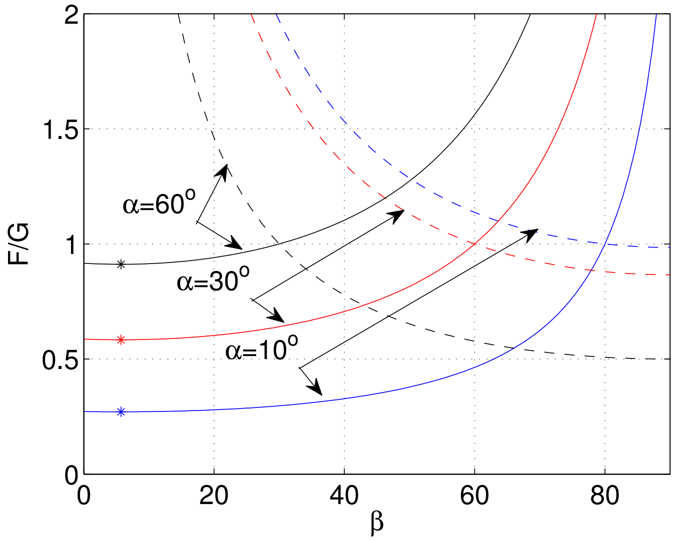

Figure 4: Solid line: the traction force needed as a function of β, see (27). Dashed line: the lift-up force given by (29). The friction coefficient is $\mu = 0.1$.

2.4 Pulley

2.4.1 Pulley without friction

A pulley, also called a sheave or a drum, is a mechanism composed of a wheel on an axle or shaft that may have a groove between two flanges around its circumference. A rope, cable, belt, or chain usually runs over the wheel and inside the groove, if present. Pulleys are used to change the direction of an applied force, transmit rotational motion, or realize a mechanical advantage in either a linear or rotational system of motion. It is one of the six simple machines. Two or more pulleys together are called a block and tackle.

The different types of pulley systems are:

Fixed A fixed pulley has a fixed axle. That is, the axle is "fixed" or anchored in place. A fixed pulley is used to change the direction of the force on a rope (called a belt).

Movable A movable pulley has a free axle. That is, the axle is "free" to move in space. A movable pulley is used to multiply forces.

Compound A compound pulley is a combination of a fixed and a movable pulley system. The *block and tackle* is a type of compound pulley where several pulleys are mounted on each axle, further increasing the mechanical advantage.

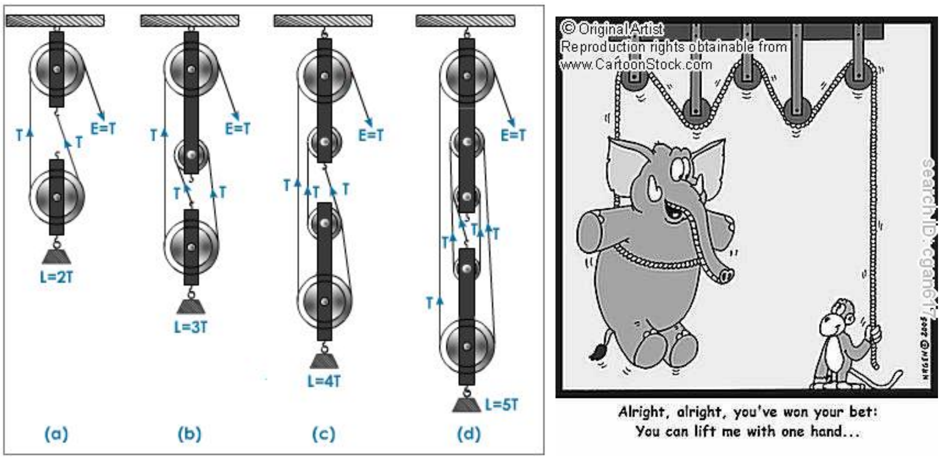

Figure 5: (Left) Pulley systems. (Right) Each student finding a mistake in the cartoon gets extra 10% in the exam.

It is important to note that as long as the friction due to sliding friction in the system where cable meets pulley and in the rotational mechanism of each pulley is neglected, the change in the potential energy of the weight $G\Delta h$ and the lifting work $F\Delta s$ are equal.

2.4.2 Pulley with friction

Figure 6 depicts a pulley with friction between the bearing and the shaft. F_1 represents the load force (in this case, one lifts a mass) while $F_{2,v}$ and $F_{2,h}$ shows two possible pulling directions. The friction force is

$$F_f = \mu N, \quad \text{where} \quad N = \begin{cases} F_1 + F_{2,v} & \text{for vertical arrangement, and} \\ \sqrt{F_1^2 + F_{2,h}^2} & \text{for vertical arrangement.} \end{cases} \tag{32}$$

The torque equilibrium is given by

$$F_2 R = F_1 R + \mu N r \tag{33}$$

In the case of vertical arrangement, this leads to

$$F_{2,v} = F_1 \frac{1 + \mu \frac{r}{R}}{1 - \mu \frac{r}{R}} \tag{34}$$

TODO: horizontal

2.5 Friction drive and belt drive

The *friction drive* (see the left-hand side of Figure 7) or friction engine is a type of transmission that, instead of a chain and sprockets, uses two

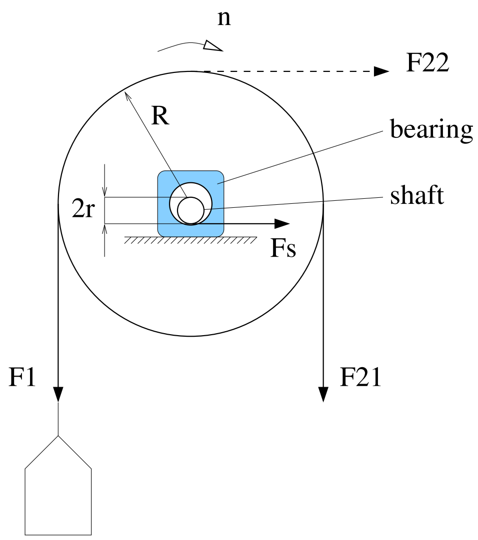

Figure 6: Pulley with friction

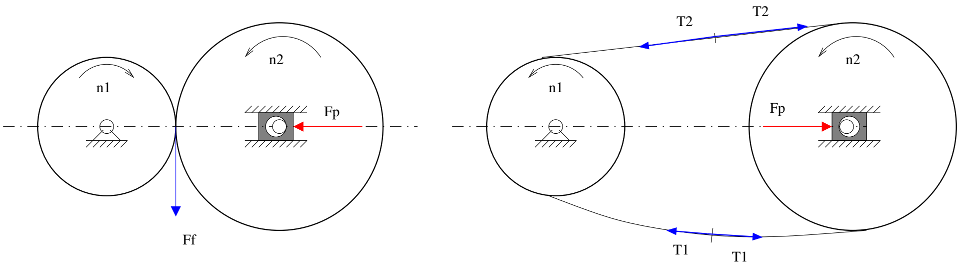

Figure 7: (left) friction drive (right) belt drive.

wheels in the transmission to transfer power to the driving wheels. This kind of transmission is often used on scooters, mainly go-peds, in place of a chain.

The friction force transmitting the power is

$$F_f = F_1 = \mu F_{cl}, \tag{35}$$

where F_{cl} is the clamping force, i.e. the force pushing the two gears against each other.

The *belt drive* (see the right-hand side of Figure 7) uses a belt, i.e. a loop of flexible material used to link two or more rotating shafts mechanically. Belts are looped over pulleys. In a two pulley system, the belt can either drive the pulleys in the same direction, or the belt may be crossed, so that the direction of the shafts is opposite. As a source of motion, a conveyor belt is one application where the belt is adapted to continually carry a load between two points.

Belt friction is a physical property observed from the forces acting on a belt wrapped around a pulley, when one end is being pulled. The equation used to model belt friction is, assuming the belt has no mass and its material is a fixed composition:

$$T_2 = T_1 e^{\mu_s \beta}, \tag{36}$$

where T_2 is the tension of the pulling side, which is typically the greater force, T_1 is the tension of the resisting side, μ_s is the static friction coefficient, which has no units, and β is the angle, in radians formed by the first and last spots the belt touches the pulley, with the vertex at the center of the pulley. The tension on the pulling side has the ability to increase exponentially if the size of the angle increases (e.g. it is wrapped around the pulley segment numerous times) and as the coefficient of friction grows. The force needed to be applied to the shaft is

$$F_{cl} = T_1 + T_2 = T_1 \left(1 + e^{\mu_s \beta}\right), \tag{37}$$

while the friction force transferring the driving torque is

$$\begin{aligned} F_f = \frac{M_1}{R_1} = T_1 - T_2 = \frac{(T_1 - T_2)(T_1 + T_2)}{T_1 + T_2} = \frac{T_1^2 \left(1 - \frac{T_2^2}{T_1^2}\right)}{T_1 \left(1 + e^{\mu_s \beta}\right)} \\ = T_1 \frac{\left(1 - e^{2\mu_s \beta}\right)}{\left(1 + e^{\mu_s \beta}\right)} = F_{cl} \underbrace{\frac{\left(1 - e^{2\mu_s \beta}\right)}{\left(1 + e^{\mu_s \beta}\right)^2}}_{\mu} \end{aligned} \tag{38}$$

Thus, for both drives, the connection between the clamping and friction force is given by $F_f = \mu F_{cl}$. •

For both drives, the *gear ratio* of the transmission is given defined as

$$i = \frac{n_1}{n_2} = \frac{\omega_1}{\omega_2}. \tag{39}$$

The input power on the driving gear is $P_1 = F_f v_1 = M_1 \omega_1$, the output power is $P_2 = F_f v_2 = M_2 \omega_2$. In the practical cases there is a slip between the driving and the driven machine:

$$s = \frac{v_1 - v_2}{v_1} = 1 - \frac{v_2}{v_1}. \tag{40}$$

The torque, revolution number and power of the *driving* machine are M_1, n_1 and P_1, respectively. On the driven side, we have

$$M_2 = F_f R_2 = F_f R_1 \frac{R_2}{R_1} = M_1 \frac{R_2}{R_1}, \tag{41}$$

$$n_2 = \frac{\omega_2}{2\pi} = \frac{v_2}{R_2 2\pi} = (1 - s)\frac{v_1}{R_1 2\pi}\frac{R_1}{R_2} = n_1(1 - s)\frac{R_1}{R_2} \quad \text{and} \tag{42}$$

$$P_2 = M_2 \omega_2 = P_1(1 - s), \tag{43}$$

with which the efficiency of the drive is

$$\eta = \frac{P_2}{P_1} = \frac{v_2}{v_1} = \frac{v_1 - (v_1 - v_2)}{v_2} = 1 - s. \tag{44}$$

2.6 Load factor, efficiency and losses of machines

The *nominal (useful) power* or *rated power* is the output power a machine was designed for. This number is fixed, usually given on the nameplate. For example, we often say that "the power of a car is 120 hp "or "a 60kW electric motor is built into the system". This is the nominal power of the machine, but it should be clear that this is not the only power the machine is capable of producing; e.g. when we sit in our car in a traffic jam with running motor, the output power is zero.

The *load factor* x is defined as the ratio of the actual useful power and the nominal power:

$$x = \frac{P_u}{P_n}. \tag{45}$$

Thus, if $P_u < P_n$, we speak about *under*load and the case when $P_u > P_n$ is called *over*load. If there is no useful work (e.g. the car is at rest but the motor runs or the computer is switched on but no user programs are running), we speak about *idle run*.

It is important to know how the efficiency of the machine depends on the load factor. The input work covers the useful work plus the losses:

$$P_{i(\text{nput})} = P_{u(\text{seful})} + P_{l(\text{oss})} \tag{46}$$

The loss consists of two parts; one being independent of the load (constant loss, P_c) and the other increasing with increasing load (variable loss, P_v):

$$P_l(x) = P_{c(\text{onstant})} + P_{v(\text{ariable})}(x) \tag{47}$$

An example of constant loss is friction of the rotating parts (e.g. bearings), which has to be covered even in idle run. An example of variable loss is the increasing heat generated as the load increases. We have

$$P_v = x^n P_{v0}, \tag{48}$$

where P_{v0} is some constant and

- $n \approx 1$ for mechanical machines (e.g. pulley)
- $n \approx 2$ for electronic machines (e.g. electric motor) and
- $n \approx 3$ for hydraulic machines (e.g. pump, fan).

The above relations allow us to give the efficiency as a function of the load factor:

$$\eta(x) = \frac{P_u}{P_i} = \frac{xP_n}{\underbrace{xP_n}_{\text{useful}} + \underbrace{P_c + x^n P_{v0}}_{\text{losses}}} \tag{49}$$

Let us find the load with maximum-efficiency:

$$\begin{aligned}
0 = \frac{d\eta(x)}{dx} &= \frac{d}{dx} \frac{\overbrace{xP_n}^{f(x)}}{\underbrace{xP_n + P_c + x^n P_{v0}}_{g(x)}} \left(= \frac{f'g - g'f}{g^2} \right) \\
&= \frac{P_n\left(xP_n + P_c + x^n P_{v0}\right) - xP_n\left(P_n + nx^{n-1}P_{v0}\right)}{\left(xP_n + P_c + x^n P_{v0}\right)^2} \\
&= \frac{P_n\left(P_c + (1-n)x^n P_{v0}\right)}{\left(xP_n + P_c + x^n P_{v0}\right)^2} \quad \rightarrow \quad P_c = (n-1)x^n P_{v0} \qquad (50)
\end{aligned}$$

Thus, we find that

for mechanical machines (n=1) there is no best-efficiency point but efficiency increases as the load is increased and $\eta \to 100\%$ as $x \to \infty$.

for electric machines (n=2) we have $P_c = x_{opt}^2 P_{v0}$, i.e. *at the best-efficiency point the constant and variable losses equal.* In other words, $x_{opt} = \sqrt{P_c/P_{v0}}$.

for hydraulic machines (n=3) we have $P_c = 2x_{opt}^3 P_{v0}$, i.e. *at the best-efficiency point the constant loss is the double of the variable loss.* In other words, $x_{opt} = \sqrt[3]{P_c/(2P_{v0})}$.

TODO: graphs

2.7 Average load and efficiency

Suppose that a machine runs at several loads during some period of time, but with constant load in each intervals. For example, from 8am to 10am x=50%, from 10am to 1pm x=90%, from 1pm to 2pm x=0% (launchtime) and finally, from 2pm to 5pm 110%. We wish to calculate the average load $\overline{x}$ and efficiency $\overline{\eta}$ during the day. Both of them will be defined with the help of work as follows:

$$\overline{\eta} = \frac{W_{\text{useful}}}{W_{\text{input}}} \quad \text{and} \quad \overline{x} = \frac{W_{\text{useful}}}{W_{\text{nominal}}} \tag{51}$$

If we have N periods and the length of the ith period is denoted by t_i and the (constant) load and efficiency is x_i and η_i, we have

$$\overline{x} = \frac{W_{\text{useful}}}{W_{\text{nominal}}} = \frac{\sum_{i=1}^{N} t_i P_{u,i}}{\sum_{i=1}^{N} t_i P_n} = \frac{\sum_{i=1}^{N} t_i x_i \cancel{P_n}}{\sum_{i=1}^{N} t_i \cancel{P_n}} = \frac{\sum_{i=1}^{N} t_i x_i}{\sum_{i=1}^{N} t_i} \tag{52}$$

and

$$\overline{\eta} = \frac{W_{\text{useful}}}{W_{\text{input}}} = \left.\frac{\sum_{i=1}^{N} t_i P_{u,i}}{\sum_{i=1}^{N} t_i P_{i,i}}\right|_{\eta_i \neq 0} = \frac{\sum_{i=1}^{N} t_i P_{u,i}}{\sum_{i=1}^{N} t_i \frac{P_{u,i}}{\eta_i}} = \frac{\sum_{i=1}^{N} t_i x_i}{\sum_{i=1}^{N} \frac{t_i x_i}{\eta_i}} \tag{53}$$

Note that the last expression of the above equation for $\overline{\eta}$ (including $t_i x_i/\eta_i$) is only valid if $\eta_i \neq 0$, i.e. there is no idle operation during the intervals!

2.8 Problems

Force components, inclined plane

Problem 2.1 A timber with mass $m = 150$kg is pulled horizontally on a flat dirty road. The pulling chain is fixed to the centre of gravity of the timber. The coefficient of friction between the the wood and the ground is $\mu = 0.48$. Find the force needed to pull the timber! ($F = 706.3$N)

Problem 2.2 Find the tractive force needed to tow a car with mass $m = 2000$kg upwards on slope whose gradient is 4% if the rolling resistance is $C_{rr} = 0.028$ and the angle between the force and the plane of the slope is $\beta = 30°$. ($F = 1515$N) Calculate the tractive power if the velocity of the towing is $v = 50$km/h. (18.22kW)

Problem 2.3 There is a ramp whose gradient is adjustable and a mass $m = 50$kg on it at rest. The slope of the ramp is slowly increased and continuously measured. It is found that at $\alpha = 7°$ the weight begins to move.

- Calculate the the coefficient of static friction and the friction force! ($\mu_s = 0.12, F_f = 59.78$N)
- Express the general relation between the angle of the slope and the coefficient of static friction.
- Find the friction force when the angel of the slope is $\alpha = 3°$. ($F_f = 25.67$N)

Problem 2.4 A car with mass $m = 1200$kg, is *pushed* upwards on a slope with constant velocity. The gradient of the slope is 6% and the angle between the pushing force ($F = 1100$N) and the horizontal direction is $\gamma = 10°$. Calculate the rolling resistance, the work performed on a 500m long distance and the part of it, that is invested in to overtake the friction! ($C_{rr} = 0.03$, $W = 534.95$kJ, $W_f = 182.39$kJ)

Friction drive

Problem 2.5 How much power can be transmitted with a friction drive, if the coefficient of friction between the wheels is $\mu = 0.37$? The diameter of the driven wheel is $D = 45$mm, its revolution number is $n = 120$rpm, and the clamping force is $F = 238$N. There is no slip between the wheels. ($P = 24.9$W)

Problem 2.6 $P = 0.4$kW power is to be transmitted with a friction drive. The wheels are supposed to roll without slip. The coefficient of friction between the wheels is $\mu = 0.4$. The diameter of the driven wheel is $D_2 = 16$cm and the applied gear ratio is $i = 2.1$ (i.e. reducing translation).

- Find the diameter of the driving wheel! ($D_1 = 7.62$cm)
- Find the clamping force (the force pushing the two fears against each other) if the revolution number of the driving shaft is $n = 2010$rpm! ($F = 124.7$N)

Belt drive

Problem 2.7 An electric motor of 14kW and 1460rpm drives a machine via a flat belt drive.

- Find the diameter of the driving pulley, if the maximal allowed belt velocity is 30m/s.
- Find the diameter of the driven pulley, if the desired translation is 2 (i.e. decelerating translation) and we are expecting 3% slip.
- Find the torque of the driven pulley.

Pulley

Problem 2.8 An electric motor drives a pulley system consisting of two standing and two moving pulleys. The system lifts a mass of 2t, the lift speed is 10 cm/s. The losses of the system can be neglected.

- Find the rope force (4.91N) and the power need of the lift ($P = 1.96$W).
- Assuming 10cm pulley radius, find the required motor torque (491Nm) and revolution number (38.2 rpm).

Load factor, losses

Problem 2.9 The efficiency of an electric generator as a function of the output power has been measured. At full load the useful power was

$P_{u,x=1} = 380\text{kW}$ while the efficiency was $\eta = 95\%$. The same efficiency was measured when the output power was 200kW. Find the constant and variable loss at full load. ($P_c = 6.9\text{kW}$, $P_{v,x=1} = 13.1\text{kW}$)

Problem 2.10 The useful power of an elevator is $P_u = 4\text{kW}$, the efficiency at this point is $\eta = 66\%$. The nominal useful power of the elevator is $P_N = 6.4\text{kW}$, while the constant loss is $P_c = 0.8\text{kW}$. The variable loss is linearly proportional to the load, i.e. the machine is mechanical. Find the variable loss at full load. ($P_{v,x=1} = 2.02\text{kW}$) Calculate the efficiency at full, half, and quarter loads. ($\eta_{x=1} = 69.4\%$, $\eta_{x=0.5} = 63.9\%$, $\eta_{x=0.25} = 55\%$)

Problem 2.11 $P_u = 140\text{kW}$ useful power is given by a transformer at full load, while its input power is $P_i = 148\text{kW}$. The transformer reaches its optimal efficiency at $x = 60\%$ load. Find the constant loss of the transformer. ($P_c = 2.12\text{kW}$) Find the optimal efficiency. ($\eta_{opt} = 95.2\%$)

Average power and efficiency

Problem 2.12 A machine repeats a *complete* working cycle periodically that lasts for $t = 40\text{min}$. During the working period the input power is $P_{i,w} = 8\text{kW}$ and the efficiency is $\eta_w = 79\%$. Between two working periods, while the work piece is replaced, the machine runs idle with a power consumption of $P_{i,i} = 1.3\text{kW}$.

- Find the time available for the replacement of the work piece if a minimum of $\eta_{avg} = 75\%$ average efficiency must be kept. ($t = 9.9\text{min}$)
- Find the average load factor if the time of the replacement reduces to $t = 8\text{min}$. ($x_{avg} = 80\%$)

Problem 2.13 A prime mover works daily for $t_1 = 4\text{h}$ at full load with an efficiency of $\eta_1 = 78\%$, $t_2 = 3\text{h}$ with a load factor of $x_2 = 0.8$ and an efficiency of $\eta_2 = 76\%$, and $t_3 = 1\text{h}$ with a load factor of $x_3 = 0.3$ and an efficiency of $\eta_3 = 56\%$. Find the average load factor and average efficiency. ($x_{avg} = 0.838$, $\eta_{avg} = 76\%$)

3 Fluid mechanics

3.1 Introduction

In many engineering systems we come across machines, which either do work on fluids (pumps, compressors, ventilators) or extract energy from fluids (water turbines or wind turbines). Examples of fluids include gases and liquids. Typically, liquids are considered to be incompressible, whereas gases are considered to be compressible. However, there are exceptions in everyday engineering applications (we shall return to this issue later). By definition, a fluid is a material continuum that is unable to withstand a static shear stress. Unlike an elastic solid which responds to a shear stress with a recoverable deformation, a fluid responds with an irrecoverable flow. The following physical quantities are of great importance:

Pressure (symbol: p, SI unit: $Pa = N/m^2 = kg/(ms^2)$) is the force per unit area applied in a direction perpendicular to the surface of an object: $p = F/A$.

Density (symbol: ρ, SI unit: kg/m^3) is defined as mass per unit volume, $\rho = m/V$. The density of a substance is the reciprocal of its specific volume ν, a representation commonly used in thermodynamics.

Compressibility is a measure of the relative volume change of a fluid as a response to a pressure change.

Ideal fluids are homogeneous, incompressible and no internal friction occurs. In many practical cases, liquids can be considered to be ideal fluids.

Real fluids are compressible. For example, for air, we know from the *ideal gas law* that $\rho = p/(RT)$, thus if air is compressed from 1 bar to 2 bar pressure by an isothermal process (i.e. temperature remains constant), its density will double. For liquids, we have $\Delta p/B = \Delta V/V$, where Δp is the pressure change, V is the initial volume, ΔV is the volume change and B is the bulk modulus. For example, for water $B_w = 2.1 \times 10^9\text{Pa} = 2.1\,\text{GPa}$ and for steel $B_s = 160\,\text{GPa}$.

Friction is also present in real fluids, which manifests itself in *pressure loss.* This simply means that e.g. the pressure along a horizontal pipe decreases in the direction of flow. This is an important issue that must be taken into account when dealing with fluid flow systems.

Pressure scales. If the term 'pressure' is used, it means absolute pressure, whose zero point is the (full) vacuum. Often we deal with relative (or gauge) pressure p_r, which is the pressure relative to the ambient pressure $p_0 = 10^5\text{Pa}$:

$$p_r = p - p_0. \tag{54}$$

Note that relative pressure can be negative, meaning vacuum. A third way or representing (vacuum) pressure is relative vacuum:

$$rel.vac. = \frac{p_0 - p}{p_0} \times 100\% \tag{55}$$

Thus 0% relative vacuum means ambient pressure p_0 and 100% relative vacuum means that $p = 0$ ($p_r = -10^5$)Pa.

TODO: graph of pressure scales.

3.2 Mass conservation - law of continuity

In fluid mechanics, mass conservation means that, under steady-state conditions, the amount of material entering and leaving a system per unit time equals. We define *mass flow rate* as the mass of substance which passes through a given surface per unit time and it can be calculated from the density of the substance, the cross sectional area through which the substance is flowing, and its velocity relative to the area of interest:

$$\dot{m} = \rho A v_{\perp}, \tag{56}$$

where $\dot{m}$ is the mass flow rate (kg/s), ρ is the density, $v_{\perp}$ is the velocity component *perpendicular to* A and A is the flow-through area. This is equivalent to multiplying the *volumetric flow rate* Q by the density:

$$\dot{m} = \rho Q, \tag{57}$$

where Q is the volumetric flow rate, its SI unit is m^3/s.

3.3 Energy conservation - Bernoulli's equation

Bernoulli's principle can be derived from the principle of conservation of energy. This states that in a steady flow the sum of all forms of mechanical energy in a fluid along a streamline is the same at all points *on that streamline*:

$$\underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}} + \underbrace{mgh}_{\text{potential energy}} + \underbrace{pV}_{\text{work done by pressure}} = \text{constant}. \tag{58}$$

However, the above equation is not useful in fluid mechanics, as it is not clear what is meant by the velocity of the fluid of mass m - the velocity changes in any small volume of fluid. Thus, we divide the above equation by V to obtain the energy content of an arbitrarily small volume:

$$\frac{\rho}{2}v^2 + \rho gh + p = \text{constant}. \tag{59}$$

Let us consider two points on a streamline (the path of a fluid particle), denoted by 1 and 2, the flow is from 1 to 2. Than, if the fluid is ideal (no friction and incompressible) and the flow is steady, we have

$$p_1 + \frac{\rho}{2}v_1^2 + \rho g h_1 = p_2 + \frac{\rho}{2}v_2^2 + \rho g h_2. \tag{60}$$

3.4 Application 1 - flow in a confuser

Consider the flow in a pipe of decreasing diameter (confuser). The diameter of the inlet is D_1, the pressure and velocity of the fluid is p_1 and v_1. The diameter of the outlet is $D_2 < D_1$, the pressure and velocity of the fluid here is p_2 and v_2.
By virtue of the continuity equation, we have

$$\rho v_1 A_1 = \rho v_2 A_2 = \rho Q, \tag{61}$$

which implies that as $A_2 < A_1$, $v_2 > v_1$. Applying Bernoulli's equation between points 1 and 2, we obtain

$$p_1 + \frac{\rho}{2}v_1^2 + \cancel{\rho g h_1} = p_2 + \frac{\rho}{2}v_2^2 + \cancel{\rho g h_2}, \tag{62}$$

from which wee see that if $v_2 > v_1$, we have $p_2 < p_1$.

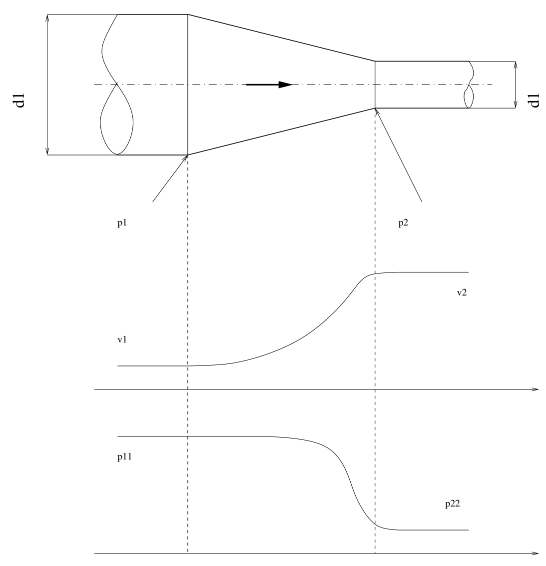

Figure 8: Flow in a confuser.

3.5 Application 2 - pressure measurement with U-tube

TODO

3.6 Problems

Problem 3.1 There is a given absolute pressure of $p = 420\text{mbar}$. Calculate the gauge pressure in pascals and the relative vacuum in percentage! ($p_g = -58\text{kPa}$, $rel.vac = 58\%$)

Problem 3.2 The plunger of a hypodermic syringe is pressed with a constant velocity of $v_1 = 0.9\text{cm/s}$. The diameter of the plunger is $D = 1.2\text{cm}$, while the radius of the hollow in the needle is $r = 0.4\text{mm}$ (see Fig.9). Find the velocity of the fluid exits at the end of the needle! ($v_2 = 2.025\text{m/s}$)

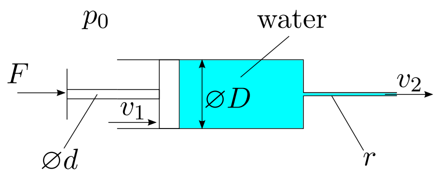

Figure 9: Sketch of Problem 3.2

Problem 3.3 Water flows through a tapering pipe with a volumetric flow rate of $Q = 115\text{dm}^3/\text{min}$. The radius at the beginning of the pipe is $r_1 = 62\text{mm}$, while the diameter at the end is $D_2 = 46.2\text{mm}$. Calculate the velocities at both ends! ($v_1 = 0.159\text{m/s}$, $v_2 = 1.143\text{m/s}$)

Problem 3.4 Air with a temperature of $T_1 = 90°\text{C}$ and a velocity of $v_1 = 15\text{m/s}$ enters into a cylindrical tube of a heat exchanger. The diameter of the tube is $d = 130\text{mm}$. At the outlet of the heat exchanger the air is cooled down to $T_2 = 20°\text{C}$. The pressure can be assumed constant along the whole tube. According to measurements the density of the air is given in the form: $\rho[\text{kg/m}^3] = 352.977/T[\text{K}]$. Calculate the velocity and the mass flow rate at the end of the tube! ($v_2 = 12.1\text{m/s}$, $\dot{m} = 0.194\text{kg/s}$)

Problem 3.5 A compressor compresses air from $p_0 = 101.325\text{kPa}$ to $p_1 = 25\text{bar}$. The mass flow rate is $\dot{m} = 30\text{kg/s}$ and the change of state is isotherm. Calculate the density at the pressure side if it is $\rho_0 = 1.2041\text{kg/m}^3$ at the suction side and is given by the function $\rho(p) = \rho_0(p/p_0)^{0.714}$! ($\rho_1 = 11.877\text{kg/m}^3$) Find the velocities and volume flow rates at the suction and pressure side if the diameters of the connected pipes are $D_0 = 900\text{mm}$ and $D_1 = 400\text{mm}$ respectively! ($v_0 = 39.16\text{m/s}$, $v_1 = 20.1\text{m/s}$, $Q_0 = 24.91\text{m}^3/\text{s}$, and $Q_1 = 2.53\text{m}^3/\text{s}$)

Problem 3.6 There is mercury with a density of $\rho_m = 13600\text{kg/m}^3$ in a U-tube manometer whose both sides are open. How high water column ($\rho_w = 1000\text{kg/m}^3$) is needed to reach a $h_m = 16\text{mm}$ difference between the levels of mercury in the two sides? ($h_w = 218\text{mm}$)

Problem 3.7 There is a U-shaped glass tube filled partially with mercury ($\rho_m = 13600$ kg/m^3). One side of the tube is opened while the other is closed. In the closed side the level of the mercury is $\Delta h = 153$mm higher than the level in the opened side. Calculate the pressure in the closed side above the mercury if the atmospheric pressure is $p_0 = 99$kPa! ($p = 78.6$kPa)

Problem 3.8 The barometer invented by Torricelli consists of a tube circa 1 meter long, sealed at the top end, filled with mercury, which is set vertically into a basin of mercury (see Fig.10). The pressure above the mercury in the tube can be neglected. Calculate the height of the mercury column ($\rho_m = 13600$kg/m^3) in the tube of the barometer on a $z = 2000$m high hill if the atmospheric pressure at the sea level is $p_0 = 101.325$kPa and the isothermal atmosphere model can be used for the pressure (i.e. $p(z) = p_0 e^{-1.166 \cdot 10^{-4} z}$ at $T = 20°$C)! ($h_m = 601$mm)

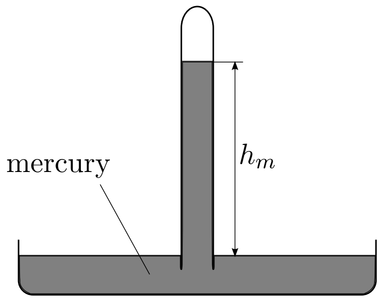

Figure 10: Sketch of Problem 3.8

Problem 3.9 Water ($\rho_w = 1000$kg/m^3) flows through a pipe with a diameter of $d = 50$mm. The volume flow rate is $Q = 72$dm^3/min and there is a height difference of $h = 3.5$m between the two ends of the pipe. The fluid flows upwards. Calculate the velocity and the pressure at the higher end if the pressure at the other end is $p_1 = 4$bar! ($v_2 = v_1 = 0.61$m/s, $p_2 = 365.7$kPa)

Problem 3.10 There is a tank filled with water ($\rho_w = 1000$kg/m^3). The top of the tank is opened to the air, while there is a small outlet at the bottom (See Fig.11). The cross sectional area of the outlet is negligible comparing to the open surface of the water. Find the velocity of the water at the outlet if the water level is $h = 2$m high! Express the velocity as a function of the height of the water level! ($v_2 = 6.26$m/s, $v_2(h) = \sqrt{2gh}$)

Problem 3.11 Water ($\rho_w = 1000$kg/m^3) flows horizontally with a velocity of $v = 5$m/s. An L-shaped tube is placed into the flow as can be seen in Fig.12. Determine difference between the level in the tube and the surface of the flow! ($h = 1.27$m/s) Calculate the gauge pressure at

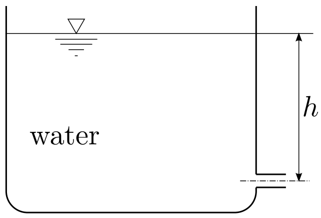

Figure 11: Sketch of Problem 3.10

the level of the leg of the tube and the stagnation point if $z = 0.5$m! ($p_{g,l} = 4.9$kPa, $p_{g,stag} = 17.4$kPa)

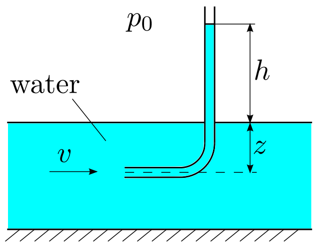

Figure 12: Sketch of Problem 3.11

Problem 3.12 In problem 3.2 calculate the force needed to press the plunger if the diameter of the rod is $d = 5$mm and the atmospheric pressure is $p_0 = 1$bar! ($F = 2.2$N)

Problem 3.13 Water ($\rho_w = 1000\text{kg/m}^3$) flows through a rising and tapering pipeline with a volume flow rate of $Q = 30\text{dm}^3/\text{s}$ (see Fig.13). The inclination of the pipeline is $\alpha = 20°$. The lower side of the pipe has a diameter of $D_1 = 200$mm, while the diameter of the upper side is $D_2 = 140$mm. A U-tube manometer filled with mercury ($\rho_m = 13600\text{kg/m}^3$) is connected to the pipeline. The length between the connection points is $l = 17$m. (The transferring fluid is also water.) Calculate the velocities and the difference between the levels of the mercury in the manometer! ($v_1 = 0.95$m/s, $v_2 = 1.95$m/s, $\Delta h = 473$mm)

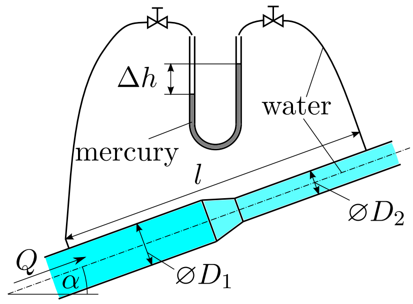

Figure 13: Sketch of Problem 3.13

4 Some basic types of machines

4.1 Internal combustion engines

The internal combustion engine (ICE) is an engine in which the combustion of a fuel (normally a fossil fuel) occurs with an oxidizer (usually air) in a combustion chamber. In an internal combustion engine the expansion of the high-temperature and -pressure gases produced by combustion applies direct force to some component of the engine, such as pistons, turbine blades, or a nozzle. This force moves the component over a distance, generating useful mechanical energy.

The term *internal combustion* engine usually refers to an engine in which combustion is intermittent, such as the more familiar four-stroke and two-stroke piston engines, along with variants, such as the Wankel rotary engine. A second class of internal combustion engines use continuous combustion: gas turbines, jet engines and most rocket engines, each of which are internal combustion engines on the same principle as previously described.

A large number of different designs for ICEs have been developed and built, with a variety of different strengths and weaknesses. Powered by an energy-dense fuel (which is very frequently petrol, a liquid derived from fossil fuels), the ICE delivers an excellent power-to-weight ratio with few disadvantages. While there have been and still are many stationary applications, the real strength of internal combustion engines is in mobile applications and they dominate as a power supply for cars, aircraft, and boats, from the smallest to the largest. Only for hand-held power tools do they share part of the market with battery powered devices.

The operation of the *four-stroke* ICE (or Otto motor) consists of four basic steps that repeat with every two revolutions of the engine:

1. Intake Combustible mixtures are emplaced in the combustion chamber

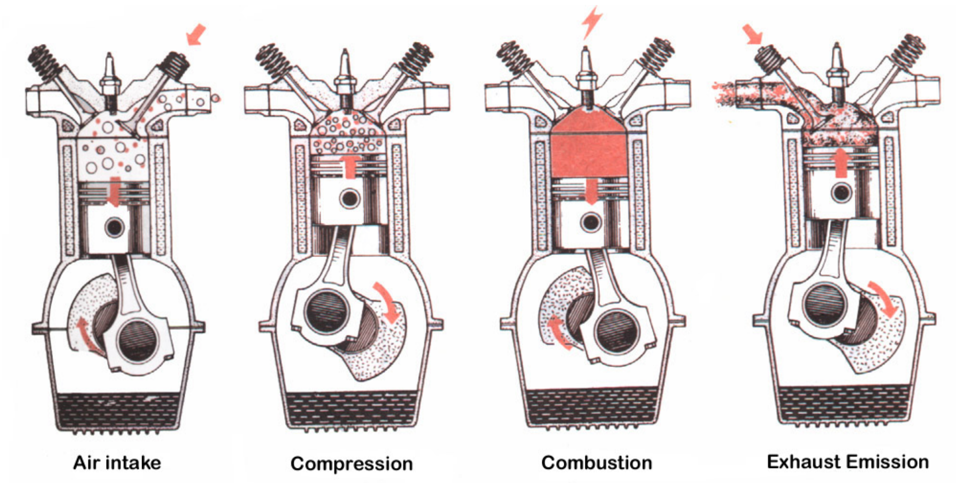

Figure 14: Operation of the Otto motor

2. Compression The mixtures are placed under pressure

3. Combustion (Power) The mixture is burnt, almost invariably a deflagration, although a few systems involve detonation. The hot mixture is expanded, pressing on and moving parts of the engine and performing useful work.

4. Exhaust The cooled combustion products are exhausted into the atmosphere.

4.2 Rankine cycle (steam engines)

A Rankine cycle describes a model of steam-operated heat engine most commonly found in power generation plants. Common heat sources for power plants using the Rankine cycle are the combustion of coal, natural gas and oil, and nuclear fission.

There are four processes in the Rankine cycle:

- Process 1-2: The working fluid is pumped from low to high pressure, as the fluid is a liquid at this stage the pump requires little input energy.
- Process 2-3: The high pressure liquid enters a boiler where it is heated at constant pressure by an external heat source to become a dry saturated vapour.
- Process 3-4: The dry saturated vapour expands through a turbine, generating power. This decreases the temperature and pressure of the vapour, and some condensation may occur.

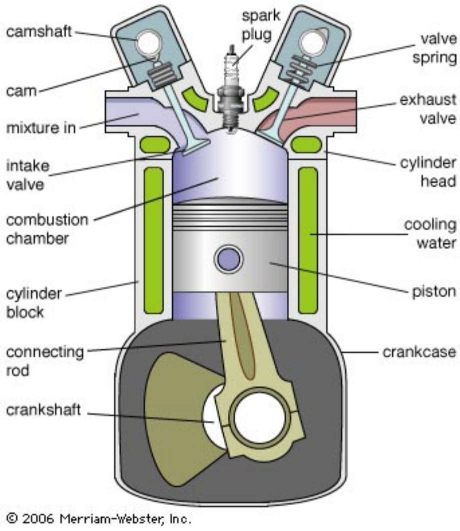

Figure 15: Elements of the Otto motor

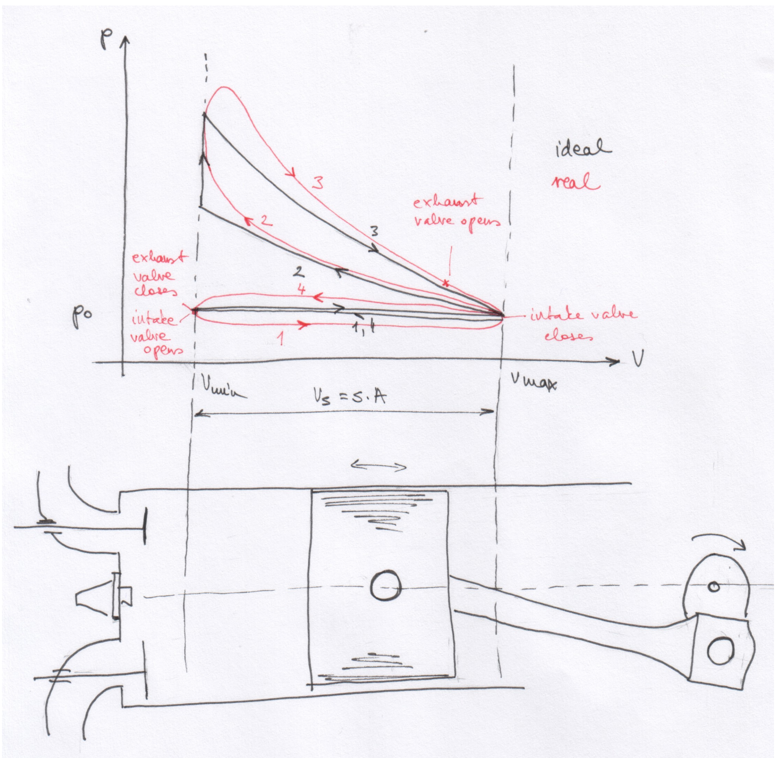

Figure 16: Indicator diagram of the Otto motor

- Process 4-1: The wet vapour then enters a condenser where it is condensed at a constant pressure to become a saturated liquid.

In an ideal Rankine cycle the pump and turbine would be isentropic, i.e., the pump and turbine would generate no loss and hence maximize the net work output.

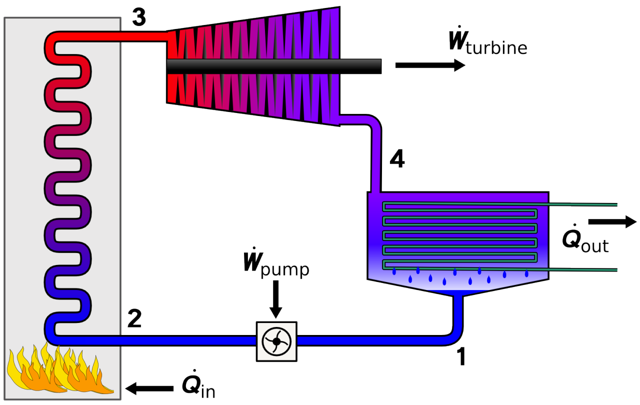

Figure 17: Rankine cycle scheme

The power balance of the cycle is as follows. Let h denote the *enthalpy* of the fluid

$$h = u + \frac{p}{\rho} + \cancel{\frac{v^2}{2}} + \cancel{gh}, \tag{63}$$

where the cancelled terms will be neglected as they are small compared to the the internal energy u and pressure work p/ρ. The internal energy u is

$$u = c_p T, \tag{64}$$

where $c_p[J/(kgK)]$ is the specific heat capacity of the fluid measured at constant pressure (for water at ambient pressure at t=15^{o}C, $c_p = 4186J/(kgK)$). The heat $\dot{Q}[J/kg]$ given to the fluid by the boiler changes the enthalpy of the fluid of mass flow rate $\dot{m}$ and is obtained by burning $\dot{m}_{fuel}$ mass flow rate fuel with heat content $H[J/kg]$:

$$\dot{Q} = \dot{m}(h_3 - h_2) = \dot{m}_{fuel} H. \tag{65}$$

The turbine extracts the output energy by decreasing the enthalpy of the fluid:

$$\dot{W}_{turbine} = P_{turbine} = \dot{m}(h_4 - h_3), \tag{66}$$

and the condenser also extracts heat from the fluid

$$\dot{Q}_{out} = \dot{m}(h_4 - h_3). \tag{67}$$

The pumping power $\dot{W}_{pump}$ is around the 1% of the turbine work output, thus it can be neglected.

4.3 Problems

Problem 4.1 During the measurement of a furnace the following data became available. The steam produced per hour is $\dot{m}_s = 6000$kg/h, while its enthalpy is $h_s = 3200$kJ/kg. The temperature of the feeding water is $t_w = 80°$C and the consumption of the coal ($H = 13.5$MJ/kg) is $\dot{m}_c = 1.6$tons/h. Calculate the efficiency of the furnace if the specific heat capacity of the water is $c_p = 4.2$kJ/(kgK) and the zero level of enthalpy is at 0°C! ($\eta = 79.6\%$)

Problem 4.2 In a steam boiler $\dot{m}_s = 2000$kg/h steam is produced with a pressure of $p_s = 10$bar and a temperature of $t_s = 250°$C from feeding water with a temperature of $t_w = 16°$C. The enthalpy of the steam is $h_s = 2940$kJ/kg, where the zero level of the enthalpy is given at 0°C and the specific heat capacity of the water is $c_p = 4.2$kJ/(kgK). The efficiency of the steam boiler is $\eta = 76\%$. Calculate the mass of coal ($H = 12$MJ/kg) needed to be burnt per hour! ($\dot{m}_c = 630$kg/h) During a reformation, the coal furnace is replaced by an oil furnace. With that change the efficiency of the steam boiler rises up to $\eta = 80\%$. Calculate the volume of the oil tank needed for a one-day-long operation if the heat content of the oil is $H = 40$MJ/kg and its density is $\rho = 950$kg/m^3! ($V = 4.536$m^3)

Worked problem 4.3 The sketch of a heat power plant can be seen in Fig. 18. The power of the electric generator is $P_{eg} = 50$MW, while its efficiency is $\eta_{eg} = 97\%$. The efficiency of the steam turbine that drives the generator is $\eta_{st} = 87\%$. The enthalpy of the freshly produced steam is $h_3 = 3370$kJ/kg, while its pressure and temperature are $p_3 = 110$bar and $t_3 = 500°$C respectively. The pressure in the condensator is $p_4 = 0.04$bar, while the enthalpy of the steam in it is $h_4 = 2180$kJ/kg.

- Calculate the mass flow rate of the steam needed to generate the given output power of the generator!
  $\dot{m}_s = \frac{P_{st,input}}{\Delta h} = \frac{P_{eg}}{\eta_{eg}\eta_{st}(h_3-h_4)} = 49.79\text{kg/s} = 179.2\text{tons/h}$
- Find the coal consumption of the steam boiler in tons/hour if its efficiency is $\eta_{sb} = 80\%$, the enthalpy of the feed water is $h_2 = 125$kJ/kg, and the heat content of the coal is $H = 20$MJ/kg!
  $\eta_{sb} = \frac{P_{sb,output}}{P_{sb,input}} = \frac{(h_3-h_2)\dot{m}_s}{\dot{m}_c H} \rightarrow \dot{m}_c = \frac{(h_3-h_2)\dot{m}_s}{\eta_{sb} H} = 36.344\text{tons/h}$
- Calculate the efficiency of the heat power plant!
  $\eta = \frac{P_{eg}}{P_{sb,input}} = \frac{P_{eg}}{\dot{m}_c H} = 24.8\%$

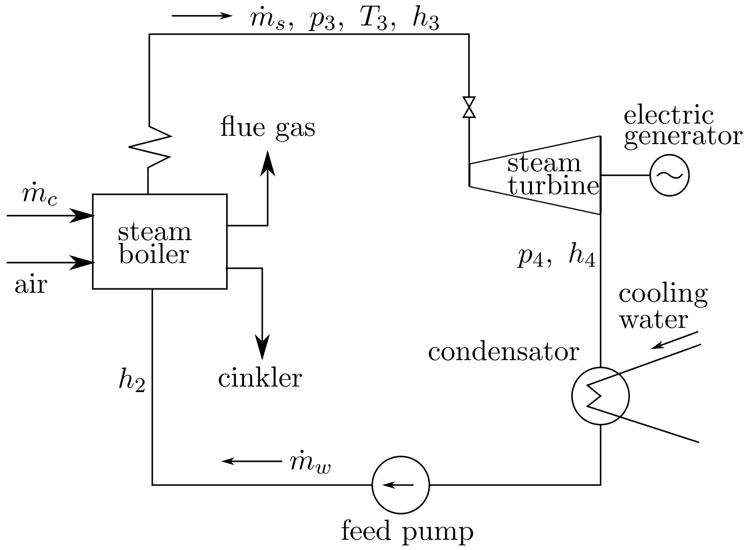

Figure 18: Sketch of the heat power plant in Problem 4.3

5 Unsteady operation of machines with constant acceleration

5.1 Introduction

Up to this point, we assumed that the forces (or torques) acting on a body are in equilibrium. Newton's second law states that

$$\sum F = ma, \tag{68}$$

that is, the acceleration of a body is parallel and directly proportional to the net force F and inversely proportional to the mass m. For rotational motion, we have

$$\sum M = \theta\varepsilon \tag{69}$$

In the engineering practice we often come across cases in which the net force $\sum F$ is constant. It follows than that the acceleration is also constant, hence the velocity and the displacement are

$$v(t) = \int_0^t a dt = at + v_0 \quad \rightarrow \quad s(t) = \int_0^t v(t) dt = \frac{a}{2}t^2 + v_0 t + s_0. \tag{70}$$

In a similar way, one obtains the following formulae for the angular velocity and rotation angle:

$$\omega(t) = \int_0^t \varepsilon dt = \varepsilon t + \omega_0 \quad \rightarrow \quad \phi(t) = \int_0^t \omega(t) dt = \frac{\varepsilon}{2}t^2 + \omega_0 t + s_0. \tag{71}$$

5.2 Examples of motion with constant acceleration

Example: Acceleration on a inclined plane

Let us reconsider the case of a body placed onto an inclined surface, as in Section 2.3, but in the absence of external force F. We have

$$ma_t = mg\sin\alpha - \mu N \neq 0 \quad \text{and} \tag{72}$$

$$ma_n = mg\cos\alpha + \underbrace{\mu N}_{F_f} = 0. \tag{73}$$

The normal acceleration is zero as the body moves parallel to the plate. Note that, on the other hand, as the friction force is unable to balance the tangential component of the gravitational force (remember that the value of μN is only the maximal possible value of the friction force!), the tangential acceleration is nonzero. The tangential acceleration is thus

$$a_t = g\left(\sin\alpha - \mu\cos\alpha\right). \tag{74}$$

Example: Braking of a rotating body

Let us consider a body with moment of inertia θ and initial revolution number $n_0 = \omega_0/(2\pi)$. We apply a constant braking force F_b on *two* brake drums, the friction coefficient is μ, the drums act on a radius R_b. We have

$$\theta\varepsilon = M_b = -2\mu F_b R_b = \text{const.} \neq 0 \tag{75}$$

The negative sign shows that we have deceleration. The angular velocity as a function of time is

$$\omega(t) = \omega_0 + \varepsilon t = \omega_0 - \frac{2\mu F_b R_b}{\theta} t, \tag{76}$$

thus the time needed for stopping the motion t_{stop} is

$$t_{stop} = \frac{\omega_0 \theta}{2\mu F_b R_b}. \tag{77}$$

The number of revolutions till the body stops is

$$N = \frac{\varphi(t_{stop})}{2\pi} = \frac{1}{2\pi}\left(\omega_0 t_{stop} - \frac{\varepsilon}{2} {t_{stop}}^2\right) \tag{78}$$

5.3 Crank mechanism

A crank is an arm attached to a rotating shaft by which reciprocating motion is imparted to or received from the shaft. It is used to change circular into reciprocating motion, or reciprocating into circular motion. The arm may be a bent portion of the shaft, or a separate arm attached to it. Attached to the end of the crank by a pivot is a rod, usually called a connecting rod. The end of the rod attached to the crank moves in a circular motion, while the other end is usually constrained to move in a linear sliding motion, in and out. The sketch of the crank mechanism is shown in Figure 19. Typical examples are:

- Mechanical pencil sharpener
- Fishing reel and other reels for cables, wires, ropes, etc.
- Manually operated car window
- The crank set that drives a trikke through its handles.
- The crankset that drives a bicycle via the pedals.
- Treadle sewing machine
- Almost all reciprocating engines use cranks to transform the back-and-forth motion of the pistons into rotary motion. The cranks are incorporated into a crankshaft.

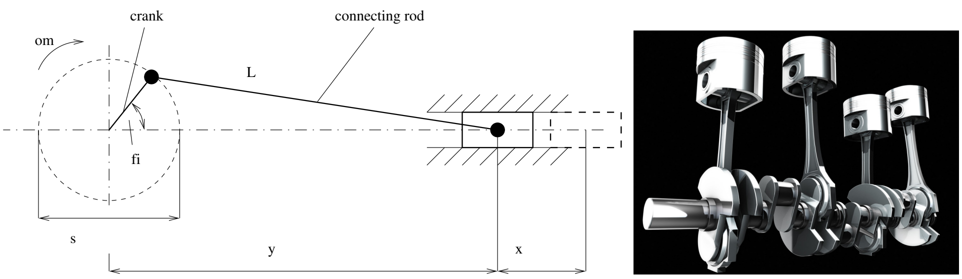

Figure 19: Sketch of the crank mechanism.

The displacement of the sliding body is

$$x(t) = L + R - y(t), \quad \text{where} \quad y = R\cos\varphi + L\sqrt{1 - \frac{R^2}{L^2}\sin^2\varphi}. \qquad (79)$$

We introduce $\lambda = R/L \ < 0.5$. The formula for displacement becomes

$$x(t) = R\left(1 - \cos\varphi - \underbrace{\frac{1}{\lambda}\left(1 - \sqrt{1 - \lambda^2\sin^2\varphi.}\right)}_{F(\lambda)}\right), \qquad (80)$$

where $\varphi = \omega t$. Now we analyze the magnitude of the function $F(\lambda)$. Note that if $\varphi = 0$, we have $F = 0$ and if $\varphi = 90^o$, $F(\lambda) = (1 - \sqrt{1-\lambda^2})/\lambda$. It is worth noting that, as shown in Table 5, if $\lambda \leq 0.2$, the error by neglecting this term is less than 10% and if $\lambda \leq 0.1$, the error is less than 5%. The displacement $x(t)$ as a function of time is shown in Figure 20. Note that the lines corresponding to $\lambda = 0.1$ and $\lambda = 0$ coincide within line width. Finally, after accepting the approx. 10% error (80) simplifies to

$$x(t) \approx R\,(1 - \cos\varphi)\,, \tag{81}$$

and the velocity and the acceleration are ($\varphi = \omega t$)

$$v(t) = R\omega\sin(\omega t) \quad \text{and} \quad a(t) = R\omega^2\cos(\omega t). \tag{82}$$

$\lambda = R/L$	0.5	0.4	0.3	0.2	0.1	0.01
$F(\lambda)$	0.2679	0.2087	0.1535	0.1010	0.0501	0.0050

Table 5: Value of function $F(\lambda) = (1 - \sqrt{1-\lambda^2})/\lambda$ for different values of λ.

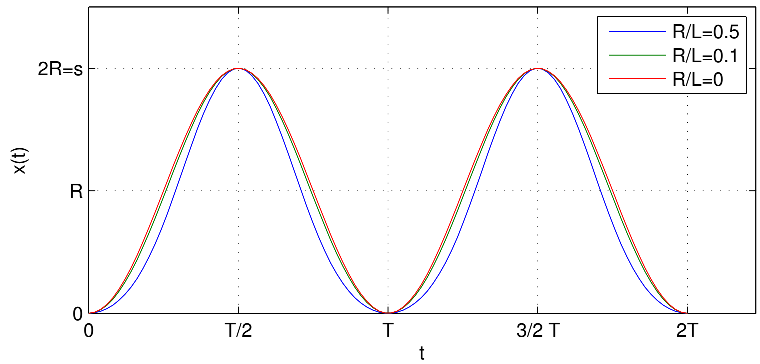

Figure 20: Displacement of the crank mechanism.

5.4 Problems

Problem 5.1 A train exhibits a traction force of 150 kN on a 600 m long path. Meanwhile, the velocity changes from 36 km/h to 54 km/h. The overall mass of the train is 1000 t. Find the change in the kinetic energy (62.5 MJ). Find the friction force (45.8 kN).

Problem 5.2 A car's velocity is 36 km/h, when it starts braking (all four wheels have brakes!). Assuming a friction coefficient of 0.6, find the diplacement needed for the stopping (8.5 m).

Problem 5.3 Due to the malfunction of the braking system, a train gets loose downwards a hill of 11 degree-slope at height 10 m. The rolling resistance is 0.08. The initial velocity was 10 m/s. Find the velocity at 0 m height (14.7 m/s).

Problem 5.4 The moment of inertia of the rotating part of a machine is 132 kgm^2. The initial revolution number is 750 rpm, the braking torque is 310 Nm. Find the time needed to stop the machine (33.4 s) and the number of revolutions (209 revs.).

Problem 5.5 The disc-like rotating part of a machine is of 0.5 m diameter and 250 kg. The initial revolution number is 1500 rpm, the friction coefficient in the bearings is 0.04, the radius of the bearing is 50mm (the radius on which the friction force acts). Find the time until the body stops if no braking force is applied (253 s). Find the braking torque needed if the element is to be stopped within 10 seconds (117.8 N).

Problem 5.6 A piston pump is driven by a crank mechanism. The radius of the crankshaft is 40mm, the revolution number is 120 rpm. The connecting rod length is large enough to assume $\lambda \approx 0$. The piston diameter is 30mm. Find the highest flow rate (0.35 liter/s) and the maximum acceleration (6.32 m/s^2).