0. measurement

THE ENGINEER MEASURES

1. Introduction

The measurement of data and their processing are the most classical ways of understanding in the field of mechanical engineering. The purpose of the first laboratory exercise in the subject of Introduction to Mechanical Engineering is to demonstrate this basic engineering method to the students who just started their studies at the Budapest University of Technology and Economics.

This classical engineering approach will be presented via the analysis of the oscillations of the well-known “mathematical pendulum”. This is probably the first time when you have to measure the properties of a physical process and draw conclusions from the collected and processed data. Because of that, we shall introduce the

- the steps of the procedure;
- the measurement itself;
- the data evaluation and graph-drawing phase;
- the analysis of the results and the understanding of possible ways of drawing a conclusion;
- and the details of the documentation.

Please note that this measurement guide, unlike further measurement descriptions, contains additional information for interested students. This additional information can be found with lower case letters and framed at the appropriate location in the relevant section. The staff of the Department of Hydrodynamic Systems wishes you success to the completion of measuring exercise.

2. The aim of the measurement

The aim of measurement is to develop empirically (i.e. based on measurements) a formula which describes the relationship of physical quantities, in order to compute the period of the oscillation.

The mathematical pendulum is a point mass ***m***, suspended on a weightless thread of length ***L*** swinging back and forth periodically under the influence of gravity.

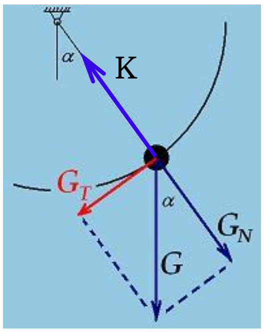
In practice, the length ***L*** of the pendulum is defined as the difference between the pivot point and the centre of gravity of swinging body. For the cylindrical weights used in our laboratory, the centre of gravity is on the longitudinal axis of the cylinder at the half of the body height (a good approximation). This place (the centre of the gravity) is marked with a circular engraved line on the surface of the weights.

The **period time** of the pendulum is defined as the time needed for the swinging body to reach the same position twice. At extremal positions, the velocity of the swinging body is zero, this is a well distinguishable and measurable position of the movement.

3. The preparation of the measurement

The measurements (both in the university, and in the industry) have large resource requirements. The measurement rig has to be built (money, time), measuring and data collecting devices have to be installed (money), and the measurement must be carried out (time), and at last the results must be processed (time). Thus, it is very important to plan the measurement in advance, and to identify the important and less important physical quantities, so that the resource need of the measurement can be reduced. Before we start the measurement, let us think about what knowledge we have about its object.

The swinging motion of the pendulum is described by the Newtonian equations of motion.

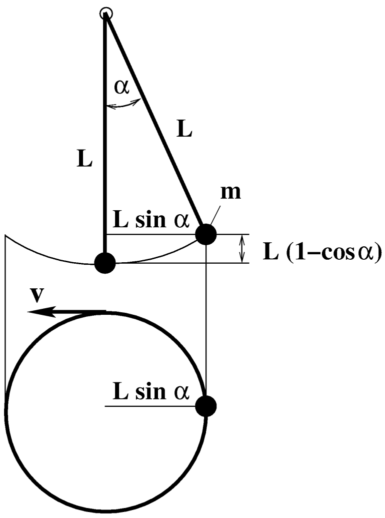


The body is accelerated by the tangential component G_T of the gravitational force that can be calculated from the weight of the body G as $G_T=G\cdot\sin(\alpha)$, where $G=m\cdot g$. With the use of the Newtonian law, the accelerating force G_T and the path component of the acceleration tangential to the path of the point mass: $m\cdot a_T=G_T$, so that

$$ma_T = mg\sin(\alpha)\,.$$

Thus we can cancel out the mass. We learnt that the **tangential acceleration and therefore the period time of the pendulum is independent of the mass of the swinging body**. This is a substantial observation the viewpoint of the measurement because it is sufficient to measure with one mass only from. We can make more measurements to check the validity of the theory, yet at the beginning of the measurement the expected results will be known. Let us just think it over, how much time we would need to “measure” this result in the laboratory, if we did not know this in advance. It is known from the secondary school studies that the equation of the period time ***T*** (1) includes the gravitational acceleration ***g***, the ***π*** and length of the pendulum ***L***:

$$T = 2\cdot\pi\cdot\sqrt{\frac{L}{g}} \qquad (1)$$



The mathematical pendulum is a point mass m, suspended on a weightless thread of length L swinging back and forth periodically under the influence of gravity. The drag coefficient and the friction at the suspension point has been neglected. The displacement is “small” (see later).
A possible derivation of the period time of the mathematical pendulum is presented. According to the **law of the energy conservation**, the sum of the pendulum's potential and kinetic energy is constant (in the lossless case). At the extreme position the pendulum has only $E_h = mgL(1-\cos\alpha)$ potential energy, and at the lowermost position it has only $E_m = mv^2/2$ kinetic energy. The two energies are equal, after the simplification with the mass and the reorganization, the maximum velocity, taken in the lowest point:

$$v = \sqrt{2gL(1-\cos\alpha)}\ .$$

Experience shows that it is a good approximation that the linear projection of the pendulum moves together with the linear projection of a point what moves along a vertical and circular path, whose radius is $L\sin\alpha$ (see figure). The angular velocity of the rotational motion is thus

$$\omega = \frac{v}{r} = \frac{\sqrt{2gL(1-\cos\alpha)}}{\sqrt{L^2\sin^2\alpha}} = \sqrt{\frac{2(1-\cos\alpha)}{\sin^2\alpha}\frac{g}{L}}\ .$$

The period time of the circular motion and the period time of the pendulum as described above:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{\sin^2\alpha}{2(1-\cos\alpha)}}\ .$$

With the introduction of $\alpha = 2\beta$ and using the $1 = \cos^2\alpha + \sin^2\alpha$ and $\cos\alpha = \cos 2\beta = \cos^2\beta - \sin^2\beta$ ***trigonometrical identities,*** subtracting from each other we get $1-\cos\alpha = 2\sin^2\beta$, and thus $\underline{2(1-\cos\alpha) = 4\sin^2\beta}$.
Another known similarity is $\sin\alpha = \sin 2\beta = 2\sin\beta\cos\beta$, and from that we get $\underline{\sin^2\alpha = 4\sin^2\beta\cos^2\beta}$.

If all of the listed equations are substituted in to the equation of the oscillation time, we get:

$$T = 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{4\sin^2\beta\cos^2\beta}{4\sin^2\beta}} = 2\pi\sqrt{\frac{L}{g}}\sqrt{\cos^2\beta} \approx 2\pi\sqrt{\frac{L}{g}}$$

The approximation can be used only when β and α ***is small*** so $\cos\beta \approx 1$. (when the angle is smaller than $\alpha = 10^\circ$ the error becomes smaller than 1.5 %.)

When a rigid body with physical extension swings around an axis that does not intersect the centre of the gravity of the body, we speak about a **physical pendulum.** The angular displacement is small in that case too and the losses can be neglected either. Physical pendulum is e.g. the system of a rotor of an electrical engine with ballast masses, or a “potato punctured with a knitting needle”.

In the equations T [s], L [m] and g [m/s²] is in the appropriate SI unit. During the measurement, one of the tasks will be the confirmation of relationship (1) between the variables, based on the measured data.
If we analyse the relationship between T and L we can see that, this is a power law function, because the equation above can be written in the form

$$T = \frac{2\pi}{\sqrt{g}}\sqrt{L} = A \cdot L^b \tag{2}$$

form, where ***A*** and ***b*** are constants. Our task is the determination of the value and the unit of ***A*** and ***b***. To reach this goal, we have to complete the measurement first!

4. The tasks of the measurement

4.1 First, by varying pendulum lengths and keeping constant swinging masses, measure the period time of the pendulum.

4.2 After that, keep constant the pendulum length, and vary the swinging masses, and measure the period time of the pendulum.

The tutor will show you the stand of the pendulum, where the threads with varying lengths can be bounded. The thread has small hooks on one side, and swivels on the other (for the connection of the stand, and for the fixation of the mass). The used thread is a braided fishing line that allows very little elongation only (it is considered rigid). Beside that, it has a small diameter and mass (both are negligible). With the fixation of the thread to the stand, and the masses to the thread, the measurement is ready to go. The person(s) who perform the measurement will use a stopwatch to measure the time needed for **at least 20 full swings** of the (max. 10°) outstretched pendulum. The count of the swings could be started with zero, at the moment when the mass is released. When the body reaches the initial position again, and starts a new swing, then we increase (even loudly) the number of oscillations (preferably: "zero-one-two-three-four-five… etc…"). It is very important to start the stopwatch exactly at the moment of the release, and to stop it precisely – after the necessary minimum number of oscillations – when it reaches again the initial position. Therefore, the same person should measure the time and start the swing. If the measurement is performed by two people, every pendulum should be measured twice with different a number of swings. The results of the measurements (where L: pendulum length, m: the mass of the swinging body, n: the counted oscillation number, *n*T*: the total swinging time) should be registered in the first four row of the following table:

	Measured values				**Calculated values**		
#	*L*	*m*	*n*	*n*T*	*T*	log*L*	log*T*
[-]	[m]	[g]	[-]	[s]	[s]	[-]	[-]
1							
2							
…							
12							

The last four column of the table will be filled out by the whole group together during the post processing phase with the help of the tutor. At the end of the measurement, the measured data should be dictated to the tutor loudly, to fill the gap in the data in a case of a possible absence.

5. Evaluation of measurement results

5.1 The determination of the coefficients

As mentioned in the Introduction, the relationship between the variables is a power function and our task is to calculate its parameters (*A* and *b*) from the measured data. To reach this goal we have 10 datasets from five different pendula. If we take the logarithm of equation (2), we get the equation of a straight line:

$$\log T = \log A + b \cdot \log L \tag{3a}$$

It is clearly seen, if we introduce the new variables $x = \log L$ and $y = \log T$, the notation $a = \log A$ then equation (3a) can be written in the form

$$y = a + b \cdot x, \tag{3b}$$

which is the equation of a straight line in the *x-y* coordinate system. After drawing the straight line, the missing coefficients (the interception point *a* and the slopeness *b*) can be determined using a graphical technique. Since only the logarithm of the variables have the property of the linear relationship ($y = \log T$ and $x = \log L$), we need to calculate and register these values in the last columns of the table of the results, and after that plot them in the same coordinate system.

During the laboratory exercise, the diagrams should be drawn with pencil on an **A4 sized plotting paper**. To make the different units distinguishable the **use of colours is recommended**.

The most important parts of the charts are the axes, with appropriate scale divisions and arrows indicating the increasing values. Besides, the markings of the scale inscriptions and the units on the axes are also essential.

The proper scale division of the diagrams is when all of the values are easily readable. It is, not practical to divide the unit into three, six, seven or nine parts. **In practice the scale division should be 1, 2 or 5 responds.** The data points on a chart should be marked as an intersection of two lines (+ sign). A marker has no distinctly identifiable characteristic point (e.g. O). The coordinates of the signed points mean the distances from the axes. The set of points which are at a constant distance from a straight line, is a straight line such parallel with the original. One point on the chart can be defined as an intersection of two straight line.

The resulting diagram will show the location of the data points. If the measurement was carried out properly, then with a good approximation the points form a straight line. A straight line should be fitted on the points with the help of a ruler, so that the number of points should be roughly equal on the two sides of the line (obviously it is not necessary to count them…), and there should not be very large and too small distances between the line and points (“the line runs among the points”). On the one hand, the intersection (a) of the drawn straight line with the y axis is easy to read, on the other hand for the calculation of the slope (b) we have to read the coordinates of two points of the straight line as far from each other as possible (x_1, y_1);(y_2,y_2).

$$B = b \cong \frac{y_2 - y_1}{x_2 - x_1}.$$

A can be calculated from:

$$A = 10^a$$

Naturally, beside the above presented graphical method there are many techniques for the fitting of the straight line to the data points. One of the most widely used techniques is the **“method of least mean squares”.** The essence of the technique is the following: we look for the minimum of the sum of the squares of the distances between the fitted line and the points.

A similar method is the so-called Wald-method, developed by Abraham Wald, who was a Hungarian-born mathematician. The technique uses centres of the gravity of the measured set of data points to determine the coefficients for the line fitting.

Beside the power function, using a proper coordinate-transformation, many functions can be linearised. For example, with the use of the above presented technique, the points of the exponential ($y=A\cdot e^{Bx}$), or the logarithm function ($y=A+B\cdot\log(x)$) can be transformed into a straight line.

5.2 The unit of the coefficients and the physical quantities contained therein

Let's assume that we have performed the measurement accurately enough to get the exponent value ***b*=½**. Knowing the concept of power, we can say, that the unit of ***b*** is one. Let us see, what the physical content of ***A*** is and let us determine them. Our equation is the following:

$$T = A \cdot \sqrt{L}$$

If we express A from the equation, we can examine what unit should ***A*** have so that the equation is dimensionally homogenous:

$$[A] = \frac{[T]}{[\sqrt{L}]} = \frac{s}{\sqrt{m}}$$

Blessed with engineering intuition, we can infer from the units that there is the following proportionality between ***A*** and gravitational acceleration:

$$A \sim \frac{1}{\sqrt{g}}$$

The known proportionality factor is 2π, so that ***A*** is defined as:

$$A = \frac{2\pi}{\sqrt{g}}$$

Let us use the value of *A* calculated from the graphical fit of *a* and estimate the magnitude of the gravitational acceleration *g* and compare it with the known value of the coefficient.

$$g = \left(\frac{2\pi}{A}\right)^2$$

(Naturally, because of the errors of the fit and the measurement, the result is just an approximation of the known value $g=9.81$ m/s^2.)

In many cases, the method above presented the so-called **"dimensional analysis"** helps in the study of a phenomenon, if we know what variables have influence on the process. This technique plays an important role if we want to define a describing equation of the phenomenon based on the measured data, or we if want to make an experimental setup for the investigation of the phenomenon. The procedure described above contains the essence of the method in a nutshell. Your later studies will certainly provide the conditions of use and the description of the exact method itself.

Recommended literature:
Simon Volker: Dimensional Analysis for Engineers, Springer International Publishing, 2017.
Franz Durst: Fluid mechanics (chapter 7 similarity theory pgs. 193-219), Springer, Berlin, 2008.

6. The discussion of the results

6.1 Based on the results of the table, it can be stated with the same thread length but with different masses, the measured period time is almost the same; the differences are caused only by obtained measurement errors.

6.2 As a result, we obtained that the formula of the period time ***T*** of a mathematical pendulum contains the gravitational field strength ***g***, ***π*** and the length of the pendulum ***L***:

$$T = 2 \cdot \pi \cdot \sqrt{\frac{L}{g}},$$

as it is known from our earlier studies. Where the units of the variables are T [s], L [m] and g [m/s^2].

6.3 So, in general, we got to know an engineering method that can be used for the analysis of many more phenomena in the future.

Preparation for measurement

- Bring yourself 1 pcs. of A4 size plotting paper, pencil, ruler and calculator.
- Fill the bianco document at home until Section 4 (5 to 8 Sections will be filled during the laboratory practice).

If you have any comment about the measurement or the lab description it is welcomed at the mail address thuzsvar@hds.bme.hu.