Impedance method

1. Basic theory

We start from the one dimensional continuity and momentum equation. The convective terms are neglected in both equations. We denote the sum of the static pressure p_{st} and hydrostatic pressure $\rho g h$ by p.

$$p = p_{st} + \rho g h ,$$

thus the basic equations are:

$$\frac{\partial p}{\partial x} + \rho \frac{\partial v}{\partial t} + \frac{\rho \lambda}{2d} v^2 = 0 , \tag{1}$$

$$\frac{1}{a^2} \frac{\partial p}{\partial t} + \rho \frac{\partial v}{\partial x} = 0 . \tag{2}$$

We consider **only periodic flows**. The average values of pressure and velocity are denoted by $\overline{p}$ and $\overline{v}$ and the periodic parts by p' and v'.

$$p = \overline{p} + p' \quad ; \quad v = \overline{v} + v' . \tag{3}$$

The average values are defined by the time integrals:

$$\overline{p}(x) = \frac{1}{T} \int_0^T p(t, x)\, dt \quad \text{and} \quad \overline{v}(x) = \frac{1}{T} \int_0^T v(t, x)\, dt . \tag{4}$$

In order to substitute (3) into (1) and (2) one has to differentiate the pressure and velocity.

$$\frac{\partial p}{\partial x} = \frac{\partial \overline{p}}{\partial x} + \frac{\partial p'}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial \overline{v}}{\partial x} + \frac{\partial v'}{\partial x} . \tag{5}$$

The derivatives with respect to time are similar. By Eqs. (4)

$$\frac{\partial \overline{p}}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \overline{v}}{\partial t} = 0 . \tag{6}$$

The average values are solutions of Eqs. (1), (2) because they represent the unperturbed case:

$$\frac{\partial \overline{p}}{\partial x} = -\frac{\lambda \rho}{2d} \overline{v}^2 \tag{7}$$

and

$$\frac{\partial \overline{v}}{\partial x} = 0 . \tag{8}$$

Further, we suppose that the velocity perturbation is much smaller than the average value: $v' << \overline{v}$. Then we can neglect v'^2 in the turbulent friction term:

$$\frac{\lambda \rho}{2d} v^2 \cong \frac{\lambda \rho}{2d} \left(\overline{v}^2 + 2 \overline{v} v' \right) . \tag{9}$$

Substituting (5) – (9) into (1) and (2) we get:

$$\frac{\partial p'}{\partial x} + \rho \frac{\partial v'}{\partial t} + \rho R v' = 0 , \tag{10}$$

$$\frac{\partial p'}{\partial t} + \rho a^2 \frac{\partial v'}{\partial x} = 0 , \text{ with} \tag{11}$$

$$R = \frac{\lambda}{d} \overline{v} .$$

The dimension of resistance R is s^{-1}.
By differentiating one of Eqs. (10) and (l1) with respect to time t, the other one with respect to x and subtracting the second from the first results in equations which contain only p' or v'.

$$a^2 \frac{\partial}{\partial x}(10) - \frac{\partial}{\partial t}(11): \qquad a^2 \frac{\partial^2 p'}{\partial x^2} - \frac{\partial^2 p'}{\partial t^2} - R\frac{\partial p'}{\partial t} = 0\,,$$

$$-\frac{1}{\rho}\left[\frac{\partial}{\partial t}(10) - \frac{\partial}{\partial x}(11)\right]: \qquad a^2 \frac{\partial^2 v'}{\partial x^2} - \frac{\partial^2 v'}{\partial t^2} - R\frac{\partial v'}{\partial t} = 0\,.$$

The similar equations are both solved by Fourier's separation method. This will result in complex pressure and velocity perturbations: Naturally, only the real part has physical meaning. ($i = \sqrt{-1}$)

$$p'(x,t) = e^{i\omega t}\left(Ae^{\gamma x} + Be^{-\gamma x}\right). \tag{12}$$

Putting this into (11) and integrating over the pipe length x we get

$$v'(x,t) = \frac{\omega}{i\rho a^2 \gamma} e^{i\omega t}\left(Ae^{\gamma x} - Be^{-\gamma x}\right). \tag{13}$$

In the above equations ω is the frequency of excitation, A and B will be determined by the boundary values, γ is called **propagation constant**. By substituting (12) and (13) into (10) we get:

$$\gamma^2 = -\frac{\omega^2}{a^2} + i\frac{R\omega}{a^2}\,.$$

As a new parameter the hydraulic impedance is introduced, it is the ratio of pressure perturbation p' and velocity perturbation v':

$$Z(x) = \frac{p'}{v'} = \frac{i\rho a^2 \gamma}{\omega} \cdot \frac{Ae^{\gamma x} + Be^{-\gamma x}}{Ae^{\gamma x} - Be^{-\gamma x}} \tag{l4}$$

The first brake is called characteristic impedance Z_c:

$$Z_c = -\frac{i\rho a^2 \gamma}{\omega} \tag{15}$$

The hydraulic impedance *Z(x)* depends only on the space coordinate. At the upstream end of the hydraulic element, $Z_u = Z(x=0)$. At the downstream end $Z_d = Z(x=L)$.

$$Z_u = Z(0) = \frac{p'(0,t)}{v'(0,t)} \tag{16}$$

$$Z_d = Z(L) = \frac{p'(L,t)}{v'(L,t)} \tag{17}$$

The coefficients A and B can be expressed with the pressure and velocity perturbation at the upstream end of the hydraulic element. Putting $x = 0$ into (12) and (13) one has

$$p'(0,t) = e^{i\omega t}(A+B) = e^{i\omega t}P_u\,,$$

$$v'(0,t) = -\frac{e^{i\omega t}}{Z_c}(A-B) = e^{i\omega t}V_u\,.$$

These give for A and B

$$A = \frac{1}{2}\left(P_u - Z_c V_u\right),$$

$$B = \frac{1}{2}(P_u + Z_c V_u)\,.$$

The perturbations over the length of the element are now:

$$p'(x,t) = e^{i\omega t}\left(P_u \cosh \gamma x - Z_c V_u \sinh \gamma x\right), \tag{18}$$

$$v'(x,t) = e^{i\omega t}\left(-\frac{P_u}{Z_c}\sinh \gamma x + V_u \cosh \gamma x\right). \tag{19}$$

Using these equations we find a relation between the upstream (P_u, V_u) and downstream values (P_d, V_d) of perturbations.

$$P_d = P_u \cosh \gamma L - Z_c V_u \sinh \gamma L \,,$$

$$V_d = -\frac{P_u}{Z_c}\sinh \gamma L + V_u \cosh \gamma L \,.$$

Using (l4)-(l6) and Eqs. (18), (19) the hydraulic impedance *Z(x)* can be written as:

$$Z(x) = \frac{Z_u - Z_c \tanh \gamma x}{1 - \frac{Z_u}{Z_c}\tanh \gamma x}\,.$$

With vector notation and putting *x* = *L* one has:

$$\begin{pmatrix} P_d \\ V_d \end{pmatrix} = \begin{pmatrix} \cosh \gamma L & -Z_c \sinh \gamma L \\ \frac{-\sinh \gamma L}{Z_c} & \cosh \gamma L \end{pmatrix} \begin{pmatrix} P_u \\ V_u \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} P_u \\ V_u \end{pmatrix} = \begin{pmatrix} \cosh \gamma L & Z_c \sinh \gamma L \\ \frac{\sinh \gamma L}{Z_c} & \cosh \gamma L \end{pmatrix} \begin{pmatrix} P_d \\ V_d \end{pmatrix}.$$

The matrix is called **impedance matrix**. The resulting impedance matrix of hydraulic elements connected in series is the product of the impedance matrices of the individual elements.
The following expression connects the impedances at the upstream and downstream ends of the element:

$$Z_d = \frac{Z_u - Z_c \tanh \gamma L}{1 - \frac{Z_u}{Z_c}\tanh \gamma L} \quad \text{and} \quad Z_u = \frac{Z_d + Z_c \tanh \gamma L}{1 + \frac{Z_d}{Z_c}\tanh \gamma L}, \quad \text{resp.}$$

2. Boundary conditions

Some simple cases are studied where either the upstream or the downstream impedance (Z_u or Z_d) can be found easily.
The pressure perturbation is zero if the **downstream pressure has a fixed constant value** (open end to the atmosphere or a liquid tank with constant surface)

$$Z_d = \frac{p'(L,t)}{v'(L,t)} = 0$$

Closed end of a pipe, the velocity perturbation is zero, thus

$$Z_d = \infty$$

Dividing or combining pipes (or other hydraulic elements) results in a boundary condition where the pressure perturbation is common for all connected elements and continuity is fulfilled. For elements *k* being connected (*k=1,2,...,K*):

$$p_1'(L_1,t) = p_2'(L_2,t) = \ldots = p_K'(L_K,t) \tag{20}$$

Supposing constant liquid density in elements having cross sections A_k gives:

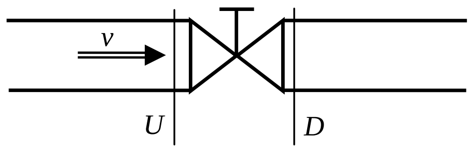
$$\sum_{k=1}^{K} \operatorname{sgn}\left(\bar{v}_k\right) \cdot A_k \left(\bar{v}_k + v_k{}'\right) = 0\,,$$

with the sign function (inflow is positive, outflow is negative). By definition $v_k' = \dfrac{p'}{Z_k}$.

Continuity must be fulfilled for the mean velocities $\bar{v}$ too thus

$$\sum_{k=1}^{K} \operatorname{sgn}\left(\bar{v}_k\right) \cdot A_k v_k{}' = 0\,. \qquad \text{or} \qquad \sum_{k=1}^{K} \operatorname{sgn}\left(\bar{v}_k\right) \cdot \frac{A_k}{Z_k} = 0\,, \tag{21}$$

The fact that the velocity is proportional to the square root of pressure difference in turbulent flow is used to formulate the impedance of a **throttle valve** in a pipe:



It is well known that

$$v = \bar{v} + v' = \mu \sqrt{\frac{2}{\rho}\left(\Delta\bar{p} + \Delta p'\right)}\,,$$

μ is the discharge coefficient.

$$\bar{v} + v' = \mu \sqrt{\frac{2}{\rho}\left(\Delta\bar{p} + \Delta p'\right)} = \mu \sqrt{\frac{2}{\rho}\Delta\bar{p}\left(1 + \frac{\Delta p'}{\Delta\bar{p}}\right)} = \mu \sqrt{\frac{2}{\rho}\Delta\bar{p}} \sqrt{1 + \frac{\Delta p'}{\Delta\bar{p}}} \approx \underbrace{\mu \sqrt{\frac{2}{\rho}\Delta\bar{p}}}_{=\bar{v}} \left(1 + \frac{\Delta p'}{2\,\Delta\bar{p}}\right)$$

From here after dropping $\bar{v}$ on both sides

$$v_u' = \frac{\bar{v}}{2\,\Delta\bar{p}}\left(p_u' - p_d'\right) \tag{22}$$

and

$$v_d' = v_u'\,. \tag{23}$$

as there is no change in the velocity perturbation through the valve.
In vector-matrix form with the conventional notations

$$\begin{pmatrix} P_d \\ V_d \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{2\,\Delta\bar{p}}{\bar{v}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P_u \\ V_u \end{pmatrix} \qquad \text{and} \qquad Z_d = Z_u - \frac{2\Delta\bar{p}}{\bar{v}}$$

The matrix has real elements thus there is no phase shift between the velocity and pressure variation through a valve.

Now we formulate the **excitation** as a boundary condition. Both velocity and pressure excitation can be handled in a similar manner. Let's see the velocity excitation. If the real velocity excitation is a sinusoidal vibration with angular frequency ω and amplitude A_o:

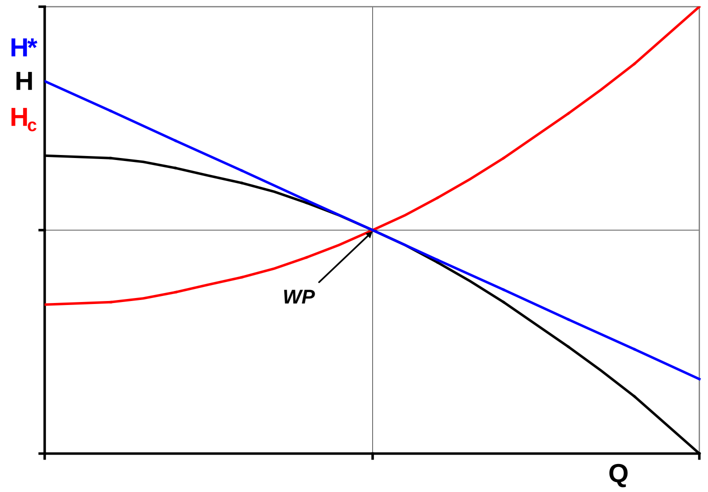
$$\operatorname{Re}\{v'\} = A_0 \sin \omega t$$

then the complex form of v' is:

$$v' = A_0 e^{i(\omega t - \frac{\pi}{2})}\,. \tag{24}$$

Really, $v' = A_0 e^{i(\omega t - \frac{\pi}{2})} = A_0 \left(\cos \omega t + i \sin \omega t\right)\left(-i\right) = -iA_0 \cos \omega t + A_0 \sin \omega t$.

As a further step we define the impedance of a **turbopump**. In the figure below we see the characteristics of the **pump** and the **system** intersecting each other in the working point. The pump characteristics may be approximated by its **tangent** in the working point.



This tangent has the equation: $H^{*} = A - BQ = A - \left.\frac{dH}{dQ}\right|_{Q=Q_{WP}} Q$. The slope of the characteristic curve in the working point can be determined graphically or numerically. Once this has been done the head and flow rate can be expressed by the pressure and velocity.

$$p_d - p_u = \bar{p}_d + p'_d - \left(\bar{p}_u + p'_u\right) = \rho g H^{*} = \rho g\left(A - BQ\right), \tag{25}$$

$$Q = \frac{D^2\pi}{4} v_u = \frac{D^2\pi}{4}\left(\bar{v}_u + v'_u\right). \tag{26}$$

Finally we suppose that the velocity doesn't change spatially through the pump: $v_d = v_u$ which holds also for the perturbed values: $v'_d = v'_u$.

From (25) and (26) we have: $\underline{\bar{p}_d - \bar{p}_u} + p'_d - p'_u = \underline{\rho g}\left(\underline{A - B\frac{D^2\pi}{4}}\left(\underline{\bar{v}_u} + v'_u\right)\right)$.

The terms underlined are equal as they represent the steady state working point. Thus; if we now denote the perturbances by upper case letters as before, we have

$$P_d - P_u = p'_d - p'_u = -\left[\rho g B \frac{D^2\pi}{4}\right]\cdot v'_u = -K\cdot v'_u = -K\cdot V_u.$$

In matrix form:

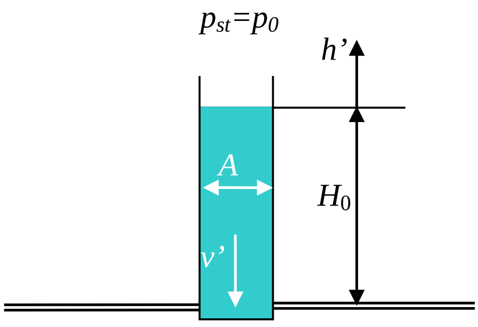
$$\begin{pmatrix} P_d \\ V_d \end{pmatrix} = \begin{pmatrix} 1 & -K \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} P_u \\ V_u \end{pmatrix} \tag{27}$$

Or the connection of impedances of a pump is

$$Z_d = Z_u - K. \tag{28}$$

Surge tank

The pressure above the free surface in the surge tank is not perturbed; it is the constant atmospheric pressure p_0. The steady state liquid height is H_0, it is perturbed, the varying water level h' is measured from H_0. The cross sectional area A of the surge tank is constant along its length. The velocity perturbation throughout the tank is v', it is equal to the derivative of h', thus $h' = -\int_0^t v' dt$.



The actual water depth in the tank is $h = H_0 + h' = H_0 - \int_0^t v' dt$ (29)

In the introduction we have defined the pressure as $p = p_{st} + \rho g h$, thus $p(x,t) = p_{st} + \rho g (H_0 + h') = p_{st} + \rho g H_0 + \rho g h' = \overline{p} + \rho g h'$. From here and from (29) we get

$$p' = -\rho g \int_0^t v' dt .$$

As $p' = Pe^{i\omega t}$ we have $\frac{dp'}{dt} = P \cdot i\omega \cdot e^{i\omega t}$. And from the above integral $\frac{dp'}{dt} = -\rho g v' = -\rho g V e^{i\omega t}$.
By combining these $P \cdot i\omega = -\rho g V$ or

$$P = i\frac{\rho g}{\omega} V ; \quad \text{from here the impedance is} \quad Z = i\frac{\rho g}{\omega} \tag{30}$$

This result shows that there is a 90° phase shift between the pressure and velocity perturbation at tank bottom.
The surge tank cross sectional are will be considered when the tank is attached to one or more pipes with much smaller cross sections, remember (21).

Example

A surge tank of A_1 = 10 m^2 cross section feeds a frictionless pipe of length L and cross section A = 1 m^2 with water.
At the downstream end of the pipe another surge tank of cross section A_2 =20 m^2 is located. Let's find the resonance frequencies of this system for varying pipe lengths. The impedance of the tank bottom is (see 30) $Z_1 = i\frac{\rho g}{\omega}$. This point is attached to the upstream end of the pipe. At this junction the flow directions are equal, thus (see 21) $\frac{A_1}{Z_1} = \frac{A}{Z_u}$, $Z_u = Z_1 \frac{A}{A_1} = i\frac{A}{A_1}\frac{\rho g}{\omega}$.
The downstream impedance of the frictionless pipe can now be computed. The propagation factor for frictionless flow is $\gamma = i\frac{\omega}{a}$; the characteristic impedance is $Z_c = -i\frac{\rho a^2}{\omega} i\frac{\omega}{a} = \rho a$.
With these (see above on page 3)

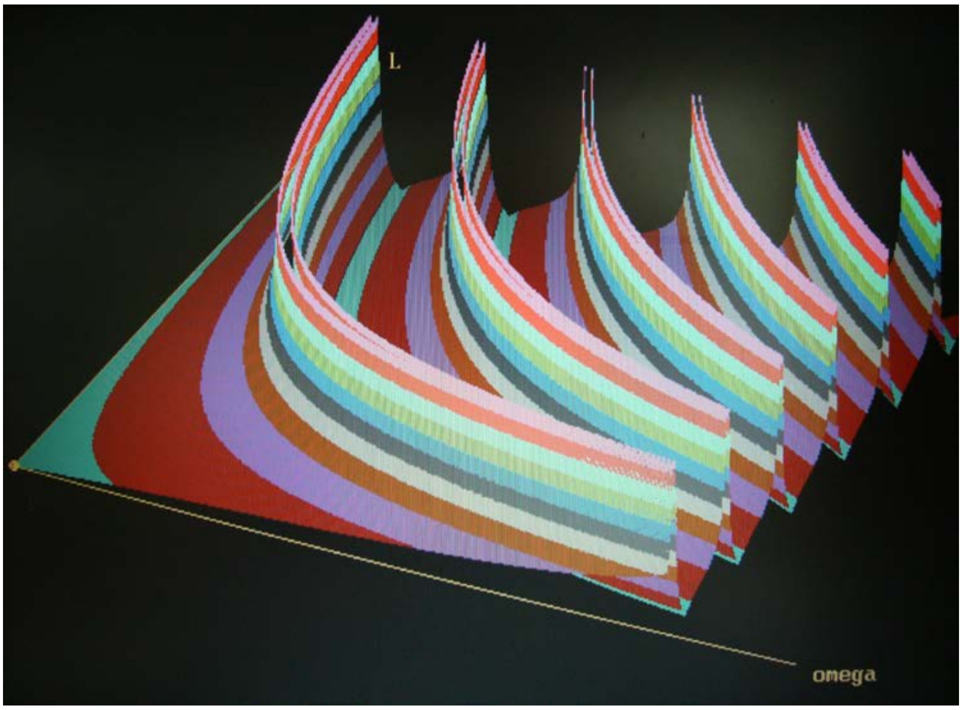
$$Z_d = \frac{Z_u - Z_c \tanh \gamma L}{1 - \frac{Z_u}{Z_c} \tanh \gamma L} = \frac{i \frac{A}{A_1} \frac{\rho g}{\omega} - \rho a \tanh\left(i \frac{\omega}{a} L \right)}{1 - i \frac{A}{A_1} \frac{\rho g}{\omega} \frac{1}{\rho a} \tanh\left(i \frac{\omega}{a} L \right)} = i \rho a \frac{\frac{A}{A_1} \frac{g}{\omega a} - \tan\left(\frac{\omega}{a} L \right)}{1 + \frac{A}{A_1} \frac{g}{\omega a} \tan\left(\frac{\omega}{a} L \right)}.$$

Here we used the identity of complex arithmetic: $\tanh(iy) = i \tan(y)$.

The impedance of the bottom of the second surge tank is now because of opposite flow directions in pipe and in tank $Z = Z_d \frac{A_2}{A}$. With this

$$|Z| = \rho a \frac{A_2}{A} \cdot \left| \frac{\frac{A}{A_1} \frac{g}{\omega a} - \tan\left(\frac{\omega}{a} L \right)}{1 + \frac{A}{A_1} \frac{g}{\omega a} \tan\left(\frac{\omega}{a} L \right)} \right| = f(\omega, L)$$

The waterfall diagram below shows the axonometric view of the $|Z|$ surface above the ω – L plane.



Waterfall diagram of a simple tank-pipe system