Steady 2D supersonic gas flow (in the diffuser part of a Laval nozzle)

The flow is completely described by the Euler equations and the continuity equation.

1st Euler: $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}=-\frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial x}=-\frac{a^2}{\rho}\frac{\partial \rho}{\partial x} \quad \Big|\cdot\left(-\frac{u}{a^2}\right)$

2nd Euler: $u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}=-\frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial y}=-\frac{a^2}{\rho}\frac{\partial \rho}{\partial y} \quad \Big|\cdot\left(-\frac{v}{a^2}\right)$

By multiplying with the given terms and adding these equations we get

$$-\frac{u^2}{a^2}\frac{\partial u}{\partial x}-\frac{uv}{a^2}\frac{\partial u}{\partial y}-\frac{uv}{a^2}\frac{\partial v}{\partial x}-\frac{v^2}{a^2}\frac{\partial v}{\partial y}=\frac{u}{\rho}\frac{\partial \rho}{\partial x}+\frac{v}{\rho}\frac{\partial \rho}{\partial y}\ . \qquad (1)$$

Continuity: $\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0=\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y} \quad \Big|:\rho$

If we divide by the ρ we get the right hand side sum of Eq. (1): $\frac{u}{\rho}\frac{\partial \rho}{\partial x}+\frac{v}{\rho}\frac{\partial \rho}{\partial y}=-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$.

Substituting this into Eq. (1) the equation of motion has been derived for the supersonic flow:

$$-\frac{u^2}{a^2}\underline{\frac{\partial u}{\partial x}}-\frac{uv}{a^2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\frac{v^2}{a^2}\underline{\underline{\frac{\partial v}{\partial y}}}+\underline{\frac{\partial u}{\partial x}}+\underline{\underline{\frac{\partial v}{\partial y}}}=$$

$$=\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial v}{\partial y}-\frac{uv}{a^2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0. \qquad (2)$$

Assuming isentropic flow the flow is rotation free too (Crocco's theorem). There exists a velocity potential $\Phi(x,y)$. The velocity components of the velocity vector ***w*** are *u* and *v* and can be derived as derivatives of the potential:

$u=\frac{\partial \Phi}{\partial x}=\Phi_x$; $v=\frac{\partial \Phi}{\partial y}=\Phi_y$. We put these and second order derivatives into Eq. (2):

$$\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx}+\left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy}-2\frac{\Phi_x\Phi_y}{a^2}\Phi_{yx}=0. \qquad (3)$$

One can prove that this is a 2nd order PDE of hyperbolic type. We introduce new independent variables ξ,η instead of *x* and *y*. Before we do it we must find an equation for the sonic velocity *a* too. In steady isentropic flow the energy equation is h_{total} = constant. The total enthalpy is:

$$h_{total}=h+\frac{w^2}{2}=c_pT+\frac{w^2}{2}=\frac{\kappa RT}{\kappa-1}+\frac{w^2}{2}=\frac{a^2}{\kappa-1}+\frac{w^2}{2}=\frac{1}{\kappa-1}\left(a^2+w^2\frac{\kappa-1}{2}\right)=\text{constant}$$

The Laval nozzle takes the air from the free atmosphere where the fluid velocity *w* is zero and the sonic speed a_0 is known. Thus

$a^2+w^2\frac{\kappa-1}{2}=a_0^2$ or by rearranging

$$a^2=a_0^2-w^2\frac{\kappa-1}{2}. \qquad (4)$$

Now the transformation of Eq. (3) must be done. In the same way as in Chapter 4 of “Unsteady flow in pipes” we differentiate Φ with respect to $x(\xi, \eta)$ and $y(\xi, \eta)$ as many times as needed.

$$\left(\left(1-\frac{u^2}{a^2}\right)\xi_x^2+\left(1-\frac{v^2}{a^2}\right)\xi_y^2-\frac{2uv}{a^2}\xi_x\xi_y\right)\frac{\partial^2\Phi}{\partial\xi^2}+(\ldots)\frac{\partial^2\Phi}{\partial\xi\partial\eta}+(\ldots)\frac{\partial^2\Phi}{\partial\eta^2}+\ldots=0\,.$$

In order to simplify this equation the first and third bracketed term must be zero, they have identical structure. The first term to be made zero us a quadratic equation for ξ_x. Solving for this we have:

$$\xi_x=\frac{\frac{2uv}{a^2}\xi_y\pm\sqrt{\frac{4u^2v^2}{a^4}\xi_y^2-4\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{a^2}\right)\xi_y^2}}{2\left(1-\frac{u^2}{a^2}\right)}=\frac{\frac{uv}{a^2}\pm\sqrt{\frac{u^2+v^2}{a^2}-1}}{1-\frac{u^2}{a^2}}\xi_y\,. \qquad (4)$$

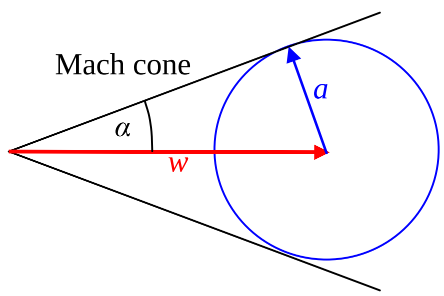
On the other hand if we consider the ξ = constant line of the new coordinate system, then its total differential is zero: $d\xi=\xi_x dx+\xi_y dy=0$ which means that $\frac{dy}{dx}=-\frac{\xi_x}{\xi_y}$. From Eq. (4) the tangent of the ξ = constant characteristic line is: $\frac{dy}{dx}=-\frac{\frac{uv}{a^2}\pm\sqrt{\frac{u^2+v^2}{a^2}-1}}{1-\frac{u^2}{a^2}}$. One sign is for ξ = constant, the other one for η = constant.

Instead of the velocity components u and v we can introduce the components of the velocity vector w,

$u=w\cos\vartheta;\quad v=w\sin\vartheta,$ further the Mach number: $M^2=\frac{w^2}{a^2}=\frac{u^2+v^2}{a^2}$. Then the tangents read

$$\frac{dy}{dx}=-\frac{M^2\sin\vartheta\cos\vartheta\pm\sqrt{M^2-1}}{1-M^2\cos^2\vartheta}\,.$$

If we perturb a supersonic flow of velocity w at some point than this perturbation propagates in the inside of the Mach cone. In 1 second the perturbed gas spot moves to a distance of w meter and the perturbation is spreading inside a circle of radius a meter. From these the sinus of the Mach cone angle α is: $\sin\alpha=\frac{a}{w}=\frac{1}{M}$.



We can write – without going into trigonometric details – that

$$\frac{dy}{dx}=\tan\left(\vartheta+\alpha\right) \text{ for } \xi=\text{constant lines and} \qquad (5)$$

$$\frac{dy}{dx} = \tan\left(\vartheta - \alpha\right) \text{ for } \eta = \text{constant lines}. \tag{6}$$

Now the flow angle ϑ can be introduced also into Eq.(2). Again without details we get

$$\frac{\partial \vartheta}{\partial \eta} - \frac{\sqrt{M^2 - 1}}{w} \frac{\partial w}{\partial \eta} = 0 \text{ for } \xi = \text{constant lines and} \tag{7}$$

$$\frac{\partial \vartheta}{\partial \xi} + \frac{\sqrt{M^2 - 1}}{w} \frac{\partial w}{\partial \xi} = 0 \text{ for } \eta = \text{constant lines and} \tag{8}$$

The computation must start just downstream of the throat of the Laval nozzle where the M number is slightly above $M = 1$. The computation may proceed inside a domain bordered by the starting vertical and two characteristic lines. Further downstream the symmetry boundary condition at the horizontal axis and the solid boundary with prescribed flow angle ϑ must be considered. The sonoc speed must be recalculated in all newly computed points from Eq. (4).

From the sonic velocity the absolute temperature $T = \frac{a^2}{\kappa R}$ can be calculated. As we have assumed isentropic flow $\frac{T}{p^{\frac{\kappa-1}{\kappa}}} = \text{constant}$ is valid thus we find the pressure distribution too and finally by the ideal gas law the density $\rho = \frac{p}{RT}$ can be computed.

The contour of a Laval nozzle downstream of the throat can be easily defined by some simple formula as e.g. $y = a - b \cdot \cos\left(x^c\right)$. The parameters must be adjusted appropriately.