Unsteady Flow in Pipe Networks

lecure notes

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Contents

1 Introductory problems

1. Write a code that counts all prime numbers up to N by means of the sieve of Eratosthenes! Check that the number of primes up to N is approximately $x/\ln x$.

2. Plot the Mandelbrot set! (If you don't know what it is, just Google it...)

3. By implementing the Runge-Kutta techniques (with embedded error estimation and variable stepsize) plot the amplification diagram of the forced 1 DoF oscillatory system

$$m\ddot{x} + k\dot{x} + sx = f_0 \sin \omega t.$$

4. Plot the Cassini oval $\left(x^2+y^2\right)^2 - 2c^2\left(x^2+y^2\right) - \left(a^4-c^4\right) = 0$ with a=1.1 and c=1 using the pseudo-arclength method!

5. Demonstrate the convergence rate of 3, 5 and 7-point finite differences for the first and second derivative on 3 different functions: $\sin(x), x \in (0, 2\pi)$, $x^2, x \in (-1, 1)$ and $\tanh(100x), x \in (-1, 1)$!

6. Given two pumps (with $Q_p = 10\, m^3/h$ constant flow rate), a suction-side reservoir and a constant ($Q_{in} = 2\, m^3/h$) inflow into the reservoir, plot the water level in the reservoir for various pump schedule policies!

7. The intensity of a line-source-like UV lamp at distance r and angle θ is $I(r, \theta) = \frac{P \cos\theta}{r^2 \pi^2}$. Using a synthetic streamline, compute the dose of the particle travelling along the streamline. The dose is $D = \int_\gamma I$, where γ is the path of the particle.

8. Using Taylor series expansion, compute the $\sqrt{x}$ at $x = 1$ with pre-defined accuracy.

2 A few numerical techniques in a nutshell

2.1 Solving systems of algebraic equtions

Consider the problem of finding the solution of

$$\underline{f}(\underline{x}) = \underline{0}, \tag{1}$$

where f is some (complicated) nonlinear function.

In the 1D case, Newton's technique (or sometime called Netwon-Raphson technique) improves the previous solution x_n by finding the intersection of the x axes and the tangent line of the function evaluated at x_n:

$$f'(x_n) \approx \frac{\Delta y}{\Delta x} = \frac{f(x_n)}{x_n - x_{n+1}} \quad \rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

In the multidimensional case, we have

$$\underline{x}_{n+1} = \underline{x}_n - \left(\underline{f}'(\underline{x}_n)\right)^{-1} \underline{f}(\underline{x}_n). \tag{3}$$

Newton's method is an extremely powerful technique, in general the convergence is quadratic: as the method converges on the root, the difference between the root and the approximation is squared (the number of accurate digits roughly doubles) at each step. However, there are some difficulties with the method.

Difficulty in calculating derivative of a function Newton's method requires that the derivative is calculated directly. An analytical expression for the derivative may not be easily obtainable and could be expensive to evaluate. In these situations, it may be appropriate to approximate the derivative by using the slope of a line through two nearby points on the function, e.g.

$$f'(x_n) \approx \frac{f(x_n + \Delta x) - f(x_n)}{\Delta x}, \quad \text{with e.g.} \quad \Delta x_n = 0.01 x_n. \tag{4}$$

Failure of the method to converge to the root Sometimes the technique runs into an infinte loop. In such cases relaxation may help, i.e. we require the method to improve the solution only partially:

$$x_{n+1} = x_n - \omega \frac{f(x_n)}{f'(x_n)}, \quad \text{with} \quad 0 \leq \omega \leq 1. \tag{5}$$

Overshoot If the first derivative is not well behaved in the neighborhood of a particular root, the method may overshoot, and diverge from that root. Furthermore, if a stationary point of the function is encountered, the derivative is zero and the method will terminate due to division by zero. Use relaxation in such cases.

Poor initial estimate A large error in the initial estimate can contribute to non-convergence of the algorithm.

Mitigation of non-convergence In a robust implementation of Newton's method, it is common to place limits on the number of iterations, bound the solution to an interval known to contain the root, and combine the method with a more robust root finding method.

Slow convergence for roots of multiplicity > 1 If the root being sought has multiplicity greater than one, the convergence rate is merely linear (errors reduced by a constant factor at each step) unless special steps are taken. When there are two or more roots that are close together then it may take many iterations before the iterates get close enough to one of them for the quadratic convergence to be apparent. However, if the multiplicity m of the root is known, one can use the following modified algorithm that preserves the quadratic convergence rate:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{6}$$

•

Note that Newton's technique is directly applicable to complex-valued functions. An interesting and beautiful experiment is to color the complex plane based on the basin of attraction of the roots of a complex polinomial function, say, $x^5 - 1 = 0$, giving rise to fractals.

The code below gives an example of Newton's method in 1D. Do some experiments with the relaxation factor `relax`.

Newton's method in 1D

```
function newtonsmethod

x=1; max_err=1e-4; max_iter=100; relax=0.5; iter=0;

f=funtosolve(x);
while abs(f)>max_err && iter<max_iter
    fprintf('\n iter=%2d, x=%7.5f, |f|=%5.3e',iter,x,abs(f));
    x=x-relax*f/derivative(x);
    f=funtosolve(x);
    iter=iter+1;
end
fprintf('\n iter=%2d, x=%7.5f, |f|=%5.3e',iter,x,abs(f));
end

function y=funtosolve(x)
y=cos(x)-x^3;
end

function dydx=derivative(x)
```

```
dydx=-sin(x)-3*x^2;
end
```

This is another example of Newton's method, now in 2D. Note that now the initial guess `x0` is a *column* vector and instead of `abs(f)`, we are using `norm(f)`.

Newton's method in 2D

```
function newtonsmethod2d

x=[1 0]'; max_err=1e-4; max_iter=100; relax=1;

f=funtosolve(x);
iter=0;
while norm(f)>max_err && iter<max_iter
    fprintf('\n iter=%2d, x=[+%7.5f +%7.5f], |f|=%5.3e',iter,x,norm(f));
    x=x-relax*inv(jac(x))*f;
    f=funtosolve(x);
    iter=iter+1;
end
fprintf('\n iter=%2d, x=[+%7.5f +%7.5f], |f|=%5.3e',iter,x,norm(f));
end

function y=funtosolve(x)
y=[cos(x(1))-x(2)^3
    x(1)*sin(x(2))];
end

function dydx=jac(x)
dydx=[-sin(x(1)) -3*x(2)^2
      sin(x(2)) x(1)*cos(x(2))];
end
```

2.2 Estimating derivatives with finite differences

2.2.1 First derivatives

Consider the problem of computing the derivative of function $f(x)$ at some point. We fix a stepsize h, than we have several possibilities: Forward difference:

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} \tag{7}$$

Backward difference:

$$f'(x) \approx \frac{f(x)-f(x-h)}{h} \tag{8}$$

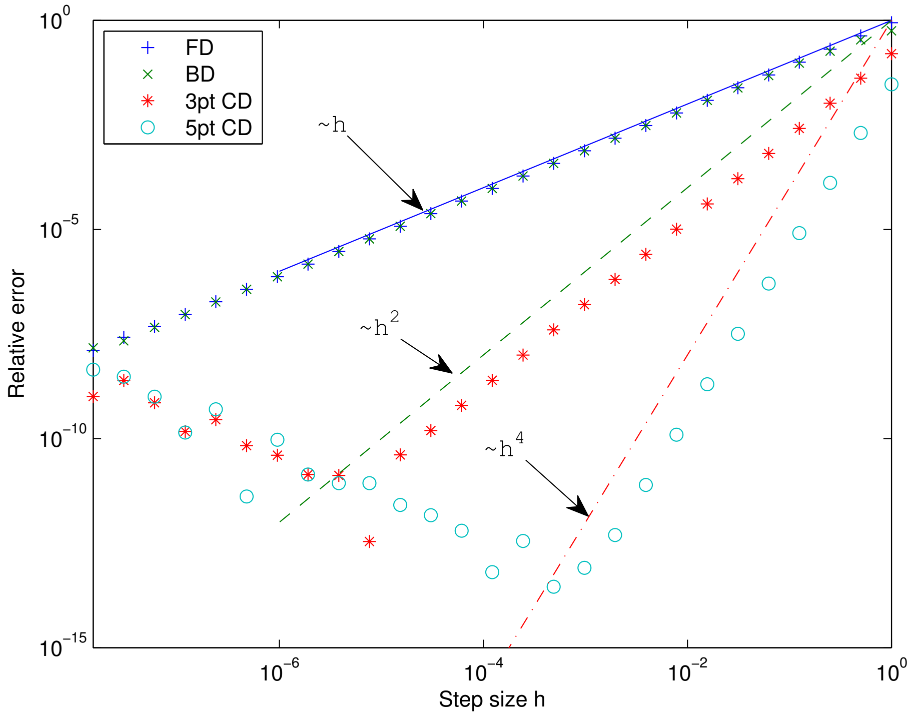


Figure 1: Accuracy of several finite difference schemes.

3-point central difference:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \tag{9}$$

5-point central difference:

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} \tag{10}$$

An important feature of a differentiation scheme is the 'speed' with which the approximation error e decreases as the step size h is decresed. An 'order m' scheme ensures that $e \propto h^m$, which is demonstrated in Figure 1. Note that one should not be greedy in terms of h as below some stepsize the limited accuracy of number representation (roundoff errors) start to spoil the accuracy of the scheme.

Numerical differentiation

```
function numdiff
figure(1), clf
h=1; x=1; dydx(1)=funder(x); e(1)=0;
for j=1:27
    dydx(2)=(f(x+h)-f(x))/h;
    dydx(3)=(f(x)-f(x-h))/h;
    dydx(4)=(f(x+h)-f(x-h))/2/h;
    dydx(5)=(-f(x+2*h)+8*f(x+h)-8*f(x-h)+f(x-2*h))/12/h;
```

```
    for i=2:length(dydx)
        e(i)=abs((dydx(1)-dydx(i))/dydx(1));
    end
    loglog(h,e(2),'+',h,e(3),'x',h,e(4),'*',h,e(5),'o'), hold on
h=h/2;
end
hh=[1 1e-6];
loglog(hh,hh,'-',hh,hh.^2,'--',hh,hh.^4,'-.'), hold off
xlabel('Step size h'), ylabel('Relative error')
legend('FD','BD','3pt CD','5pt CD',2), axis([0 1 1e-15 1])
end

function y=f(x)
y=sin(x);
end

function dydx=funder(x)
dydx=cos(x);
end
```

2.2.2 Second derivatives

The simplest possibility is to use the central difference scheme at the half grid points (second-order accuracy):

$$\begin{aligned} f''(x) &\approx \frac{f'\left(x+\frac{h}{2}\right)-f'\left(x-\frac{h}{2}\right)}{\Delta h} = \frac{1}{h}\left(\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}\right) \\ &= \frac{f(x+h)-2f(x)+f(x-h)}{h^2} \end{aligned} \tag{11}$$

Another scheme using five points providing fourth-order accuracy is

$$f''(x) \approx \frac{-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)}{12h^2}. \tag{12}$$

2.3 Solving ordinary differential equations

In general a system of ordinary differential equations (ODEs) have the form of

$$\underline{x}' = \underline{f}\left(\underline{x}, t\right), \tag{13}$$

where the function $\underline{x}(t)$ is to be determined. In the case when the function $\underline{f}$ is only dependent upon t implicitly, that is $\underline{f} = \underline{f}(\underline{x})$, the ODE is called *autonomous*, otherwise it is called *non-autonomous*. There are two main groups of problems that can be described by ODEs:

- Boundary value problems (BVPs), where the function values at the boundaries ($\underline{x}(a)$ and $\underline{x}(b)$) are prescribed. These are usually describing a problem where the independent variables are space coordinates.

- Initial value problems (IVPs), where the function values are given at an initial time ($\underline{x}(t_0)$). These are usually describing problems where the independent variable is time. (In this subject we mainly deal with IVPs.)

In most of the cases there is no known analytical solution for an IVP. In these cases one tries to find a numerical approximation for the solution function ($\underline{x}(t)$). All these numerical methods are common in the sense that they give an approximate formula for the time derivative ($\dot{\underline{x}}$) and calculate the approximate solution at distinct time steps ($\underline{x}_n(t_n)$). Here three basic methods will be presented briefly.

2.3.1 Explicit Euler method

The basic equation reads:

$$\dot{\underline{x}} \approx \frac{\underline{x}(t+\Delta t) - \underline{x}(t)}{\Delta t} = \frac{\underline{x}_{n+1} - \underline{x}_n}{\Delta t} = \underline{f}(\underline{x}_n, t_n), \tag{14}$$

thus in every new time step the new function value can be computed by the formula:

$$\underline{x}_{n+1} = \underline{x}_n + \Delta t \underline{f}(\underline{x}_n, t_n). \tag{15}$$

This method is explicit since the new function value ($\underline{x}_{n+1}$) can be explicitly expressed with the help of the previous values ($\underline{x}_n$), hence it is fast. On the other hand it is very unstable and inaccurate (first order method), so in general it is not used. One way to improve the method is to use implicit Euler method.

2.3.2 Implicit or backward Euler method

The basic equation reads:

$$\dot{\underline{x}} \approx \frac{\underline{x}_{n+1} - \underline{x}_n}{\Delta t} = \underline{f}(\underline{x}_{n+1}, t_{n+1}). \tag{16}$$

This method is implicit, since for every new time step one has to solve a system of algebraic equations to get $\underline{x}_{n+1}$. This can be performed for example by the previously presented Newton's method. One advantage of the method is its stability (see Section 2.3.4), however its accuracy is not improved (first order method). One way to develop the accuracy is to use a higher order scheme such as Runge-Kutta 4.

2.3.3 Runge-Kutta 4 method

The basic equation reads:

$$\underline{x}_{n+1} = \underline{x}_n + \Delta t \left[\frac{\underline{k}_1}{6} + \frac{\underline{k}_2}{3} + \frac{\underline{k}_3}{3} + \frac{\underline{k}_4}{6} \right], \tag{17}$$

with

$$\begin{aligned}
\underline{k}_1 &= \underline{f}\left(\underline{x}_n, t_n\right), \\
\underline{k}_2 &= \underline{f}\left(\underline{x}_n + \frac{\Delta t}{2}\underline{k}_1, t_n + \frac{\Delta t}{2}\right), \\
\underline{k}_3 &= \underline{f}\left(\underline{x}_n + \frac{\Delta t}{2}\underline{k}_2, t_n + \frac{\Delta t}{2}\right), \\
\underline{k}_4 &= \underline{f}\left(\underline{x}_n + \Delta t \underline{k}_3, t_n + \Delta t\right).
\end{aligned}$$

This method is fourth order as it is denoted in its name. This means a great improvement in its accuracy over the Euler method. Since it is an explicit scheme it is fast as well.

2.3.4 Stability

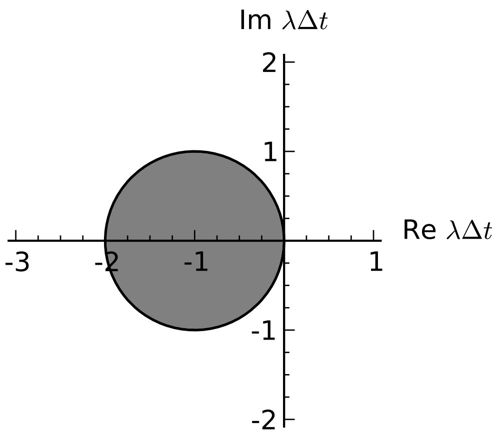
Stability is a very important property of the different kinds of ODE solvers. The common way of investigating a method's stability is to consider the following scalar ODE:

$$\dot{x} = \lambda x, \quad \text{where} \quad \lambda \in \mathbb{C} \quad \text{and} \quad \Re(\lambda) < 0. \tag{18}$$

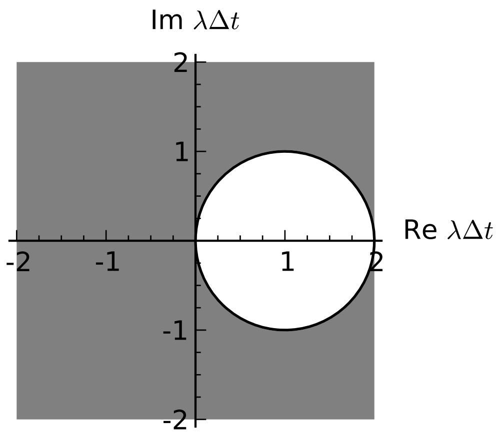
The condition $\Re(\lambda) < 0$ is required since we want to deal with a 'normal physical phenomenon' that is stable and only the numerical method can introduce instability in the solution. If we apply the different schemes to (18) we get the following geometric series:

- Explicit Euler method: $x_{n+1} = (1 + \Delta t\lambda)x_n$
- Implicit Euler method: $x_{n+1} = \frac{1}{1-\Delta t\lambda}x_n$
- Runge-Kutta 4 method: $x_{n+1} = \left(\Delta t\lambda + \frac{(\Delta t\lambda)^2}{2!} + \frac{(\Delta t\lambda)^3}{3!} + \frac{(\Delta t\lambda)^4}{4!}\right) x_n$

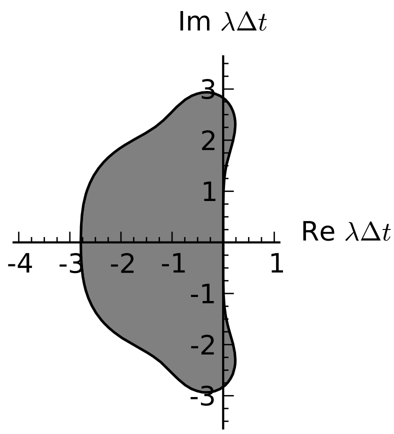
Since a geometric series is stable if $|q| < 1$, where q is the quotient, we have to investigate the coefficient of x_n. In Figure 2 the stability regions of the different ODE solvers are plotted. Since $\Re(\lambda) < 0$ and $\Delta t \in \mathbb{R}^+$, $\Re(\Delta t\lambda) < 0$, that is we are only interested in the left side of the complex plane. Δt should be chosen so that $\Delta t\lambda$ is in the stability regions. As it can be seen the implicit Euler method is unconditionally stable, while the stability region of the Runga-Kutta 4 method is wider than the explicit Euler's.



(a)



(b)



(c)

Figure 2: Stability charts of the explicit Euler (a), the implicit Euler (b) and the Runge-Kutta 4 (c) methods (stable regions are *gray*).

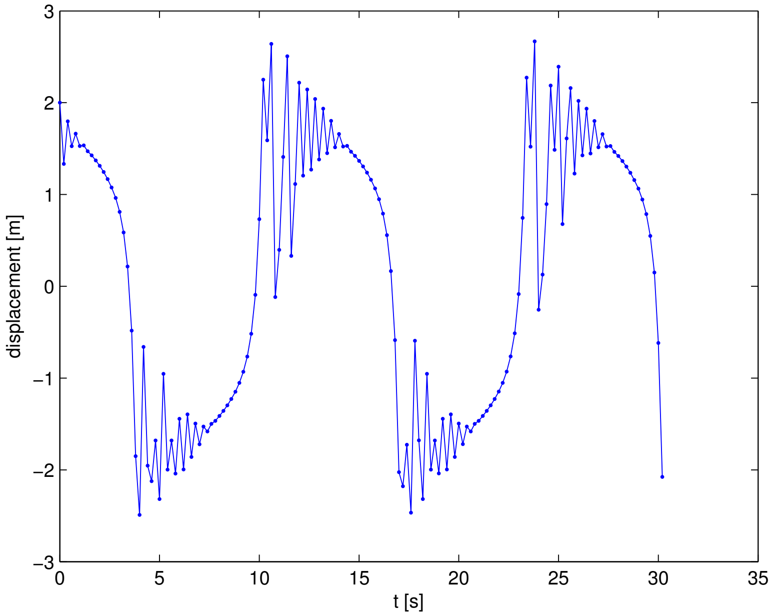
2.3.5 Accuracy

Another very important aspect of ODE solvers is accuracy. Basically there are two ways to improve accuracy: using a higher order method (such as Runge-Kutta 4) or using a so-called adaptive method (such as Runge-Kutta-Fehlberg 45). The basis of an adaptive method is to somehow estimate the error of each new step and adopt the time-step so that a prescribed error tolerance is kept. A very simple way to achieve that (although computationally very expensive) is to determine the new value by two different ways: perform one step with the time-step from the previous step (Δt) and stepping two steps with $(1/2)\Delta t$, then comparing the two values. If the difference of the two values are greater than a prescribed error, than the time-step is decreased and the method is repeated, otherwise the new value is accepted and we continue on. Such a method is implemented below for the explicit Euler scheme on the Van der Pol equation. The Van der Pol equation is often used to test adaptive solvers, since it is stiff, that is the solution function has very rapid variations. The output of the scheme is shown in Figure 3. (It is worth noting that an adaptive solver poses a solution to instability issues as well.)

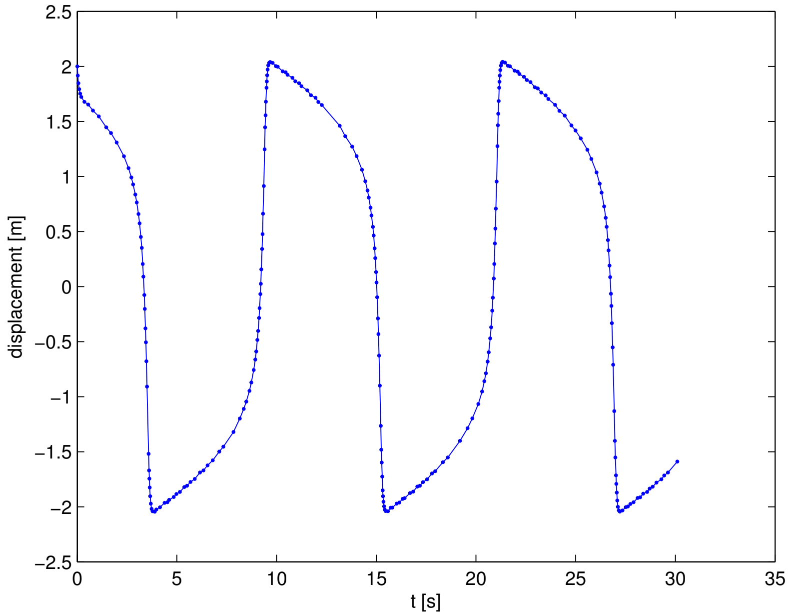
Adaptive explicit Euler method for the Van der Pol equation

```
function van_der_pol

max_err=1e-2;       % maximum allowed error
t0=0;               % initial time
dt=0.2;             % initial time step
tEnd=30;            % final time
x0=[2,0];           % initial condition
adap=1;             % adaptivity switcher
inc_fac=5;          % increment factor for dt

t=t0; x=x0; tVec=t0; xVec=x0;
if (adap==1)    % adaptivity on
    while (t < tEnd)
        xNew1=x+dt*ode_fun(x);       % one step with dt
        xNew2_1=x+dt/2*ode_fun(x);   % two steps with dt/2
        xNew2=xNew2_1+dt/2*ode_fun(xNew2_1);
        if (norm(xNew1-xNew2) < max_err)
            tVec=[tVec, t+dt]; xVec=[xVec; xNew1];
        else
            % halving dt till required error is reached
            while (norm(xNew1-xNew2) > max_err)
                dt=dt/2;
                xNew1=x+dt*ode_fun(x);
                xNew2_1=x+dt/2*ode_fun(x);
                xNew2=xNew2_1+dt/2*ode_fun(xNew2_1);
            end
            tVec=[tVec, t+dt]; xVec=[xVec; xNew1];
```

```
        end
        t=t+dt; x=xNew1; dt=dt*inc_fac;
    end
elseif (adap==0) % adaptivity off
    while (t < tEnd)
        xNew=x+dt*ode_fun(x);
        tVec=[tVec, t+dt]; xVec=[xVec; xNew];
        t=t+dt; x=xNew;
    end
end

plot(tVec,xVec(:,1),'.-'); xlabel('t [s]'); ylabel('displacement [m]')

function dxdt=ode_fun(x) % Van der Pol equation with mu=5
dxdt(1)=5*(x(1)-1/3*x(1)^3-x(2));
dxdt(2)=1/5*x(1);
```



(a)



(b)

Figure 3: Computed displacement of the Van der Pol equation with regular (a) and with adaptive (b) solvers

3 Unsteady 1D slightly compressible fluid flow

3.1 Governing equations

Let us start with the 1D incompressible equation of motion and continuity equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = S(x,t) \quad \text{where} \quad S(x,t) = -g\frac{dz}{dx} - \frac{\lambda}{2D}|v|v + a_x, \tag{19}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0 \tag{20}$$

Here $S(x,t)$ is the source term, including acceleration due to the inclination of the pipe ($g\, dz/dx$), friction and possible acceleration in the horizontal direction a_x.

Let us assume that the fluid is barotropic, i.e. $\rho = \rho(p)$. The continuity equation turns into

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = \underbrace{\frac{d\rho}{dp}}_{a^{-2}}\frac{\partial p}{\partial t} + \rho\frac{\partial v}{\partial x} + v\frac{d\rho}{dp}\frac{\partial p}{\partial x} = \frac{1}{a^2}\left(\frac{\partial p}{\partial t} + \rho a^2\frac{\partial v}{\partial x} + v\frac{\partial p}{\partial x}\right) = 0, \tag{21}$$

where a is the sonic velocity. Now we calculate a^2(21)+ρa (19):

$$\frac{\partial\left(p + \rho a v\right)}{\partial t} + (v+a)\frac{\partial\left(p + \rho a v\right)}{\partial x} = \rho a S(x,t). \tag{22}$$

Computing a^2(21)-ρa(19) gives

$$\frac{\partial\left(p - \rho a v\right)}{\partial t} + (v-a)\frac{\partial\left(p - \rho a v\right)}{\partial x} = -\rho a S(x,t). \tag{23}$$

Note that the above derivatives are directional derivates:

$$\frac{\mathcal{D}^+\clubsuit}{\mathcal{D}t} := \frac{\partial\clubsuit}{\partial t} + (v+a)\frac{\partial\clubsuit}{\partial x} \quad \text{is a derivative along the line} \frac{dx}{dt} = v+a$$

$$\frac{\mathcal{D}^-\clubsuit}{\mathcal{D}t} := \frac{\partial\clubsuit}{\partial t} + (v-a)\frac{\partial\clubsuit}{\partial x} \quad \text{is a derivative along the line} \frac{dx}{dt} = v-a$$

Hence, by defining $\alpha = p + \rho a v$ and $\beta = p - \rho a v$, the *ordinary differential equations* to be solved are

$$\frac{\mathcal{D}^+\alpha}{\mathcal{D}t} = \rho a S(x,t) \quad \text{and} \quad \frac{\mathcal{D}^-\beta}{\mathcal{D}t} = -\rho a S(x,t). \tag{24}$$

3.2 Application on pressurized liquid pipeline systems

In the case of pressurized liquid pipelines (e.g. water distribution systems or oil pipeline systems) the flow velocity in the pipeline v is typically in the range of a few m/s while the wave velocity a is in the range of $1000\, m/s$. The bulk modulus of water is $B = 2.1$GPa, which gives $a = \sqrt{B/\rho} = \sqrt{2.1 \times 10^3} \approx 1400\, m/s$. Hence the slope of te characetristic lines $v \pm a$ is hardly effected by the fluid velocity. The assumption $v \ll a$ allows the usage of a fix grid as the characteristic slopes are constant: $dx/dt = \pm a$, as shown in Figure 4.

3.2.1 Update of the internal points

A simple numerical scheme can be built onto (24), see Figure 4. We use a special grid that strictly satisfies

$$\frac{\Delta x}{\Delta t} = a, \tag{25}$$

meaning that if e.g. the spatial grid is set, the time step cannot be chosen arbitrarily but must be computed based on (25). The first step updates the α and β values in the internal points

$$\frac{\alpha_j^{i+1} - \alpha_{j-1}^i}{\Delta t} = \rho a S_{j-1}^i \quad \rightarrow \quad \alpha_j^{i+1} = \alpha_{j-1}^i + \Delta t \rho a S_{j-1}^i \quad j = 2 \ldots N-1 \tag{26}$$

$$\frac{\beta_j^{i+1} - \beta_{j+1}^i}{\Delta t} = -\rho a S_{j+1}^i \quad \rightarrow \quad \beta_j^{i+1} = \beta_{j+1}^i - \Delta t \rho a S_{j+1}^i \quad j = 2 \ldots N-1. \tag{27}$$

Then, we compute pressure and velocity simply by

$$p = \frac{\alpha + \beta}{2} \quad \text{and} \quad v = \frac{\alpha - \beta}{2\rho a}. \tag{28}$$

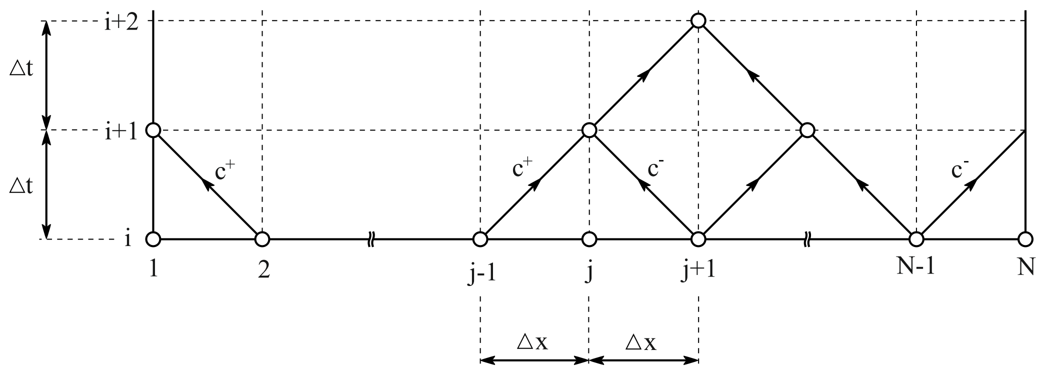


Figure 4: Incompressible MOC, numerical scheme.

3.2.2 Boundary conditions

At the boundary points we have only one characteristic equation: on the left boundary $j = 1$ based on (27) we have

$$p_1^{i+1} - \rho a v_1^{i+1} = p_2^i - \rho a v_2^i - \Delta t \rho a S_2^i := K_l. \tag{29}$$

On the right boundary $j = N$, based on (26) we have

$$p_N^{i+1} + \rho a v_N^{i+1} = p_{N-1}^i + \rho a v_{N-1}^i + \Delta t \rho a S_{N-1}^i := K_r \tag{30}$$

It is clear that the above equations include two unknowns, that is $(p_{1,N})^{i+1}$ and $(v_{1,N})^{i+1}$. The second equation required to obtain a solution is provided by the actual boundary condition as follows.

We shall give the equations for the left boundary only, the right-hand end can be treated in a similar way.

Prescribed velocity This case applies to both closed pipe or prescribed consumption/flow. As v_1^{i+1} is known ($v_1^{i+1} = Q_{in}/A_{pipe}$) (assuming inflow to be positive) thus p_1^{i+1} can be computed based on (29).

Prescribed pressure. p_1^{i+1} is known, thus v_1^{i+1} can be computed based on (29).

Prescribed total pressure: $p_1^{i+1} + \frac{\rho}{2}\left(v_1^{i+1}\right)^2$ is known and must be solved together with (29).

For example, in pool type cases for inflow, we have

$$p_0 + \rho g H = p_1^{i+1} + \frac{\rho}{2}\left(v_1^{i+1}\right)^2 \tag{31}$$

For backflow we have

$$p_0 + \rho g H = p_1^{i+1} \tag{32}$$

We have to decide between inflow and outflow as follows.

$p_0 + \rho g H := P_0 = p_1^{i+1} + \frac{\rho}{2}\left(v_1^{i+1}\right)^2$ is known and must be solved together with (29).

From now on, $v = v_1^{i+1}$ and $p = p_1^{i+1}$

We have only one physical solution for the velocity, which must be greater than zero in case of inflow.

$$v = -a + \sqrt{a^2 - \frac{2}{\rho}\left(K_l - P_0\right)} \tag{33}$$

We obtain the result that in case of inflow: $K_l < P_0$

Pipe junction: M pipes mean $M+1$ unknowns, what we can obtain from the equations of incoming pipes: $p + \rho a v_i = K_{ri}$, of outgoing pipes: $p - \rho a v_i = K_{li}$, and the plus one equation from continuity: $\sum_{i \epsilon in} v_i A_i = \sum_{i \epsilon out} v_i A_i + C$, where C means the consumption.

From these equations we obtain:

$$\sum_i \frac{K_i - p}{\rho a} - C = 0 \tag{34}$$

$$\sum_i \frac{A_i K_i}{\rho a} - p \sum_i \frac{A_i}{\rho a} - C = 0 \tag{35}$$

$$p = \frac{\sum_i \frac{A_i K_i}{\rho a} - C}{\sum_i \frac{A_i}{\rho a}} \tag{36}$$

Pump: The operation of a pump is described by power input (P), flow rate (Q) and head (H) at different revolution numbers (n). The head curve (head versus flow rate) and the input power curve (input power versus flow rate) at a given n_0 revolution number can be seen on Figure 5.

The connection between Q, H, P and n is prescribed by the affinity laws:

$$\frac{Q}{Q_0} = \frac{n}{n_0} \tag{37}$$

$$\frac{H}{H_0} = (\frac{n}{n_0})^2 \tag{38}$$

$$\frac{P}{P_0} = (\frac{n}{n_0})^3 \tag{39}$$

$$\frac{P}{P_0} = \frac{\rho g H Q}{\rho g H_0 Q_0} \tag{40}$$

In the catalogue the head (H_0) and the input power (P_0) is given as a function of the flow rate (Q_0):

$$H_0(Q_0) = a_0 + a_1 Q_0 + a_2 Q_0^2 + a_3 Q_0^3 + ... \tag{41}$$

$$P_0(Q_0) = b_0 + b_1 Q_0 + b_2 Q_0^2 + b_3 Q_0^3 + ... \tag{42}$$

With the help of affinity laws these equations can be rewritten and the head (H) and the input power (P) can be determined at any revolution number (n):

$$H_0 = H(\frac{n_0}{n})^2 \tag{43}$$

$$H(\frac{n_0}{n})^2 = a_0 + a_1 Q \frac{n_0}{n} + a_2 Q^2 (\frac{n_0}{n})^2 + a_3 Q^3 (\frac{n_0}{n})^3 + ... \tag{44}$$

$$H(Q, n) = a_0(\frac{n}{n_0})^2 + a_1(\frac{n}{n_0})Q + a_2 Q^2 + a_3 Q33(\frac{n_0}{n}) + ... \tag{45}$$

$$P(Q,n) = b_0(\frac{n}{n_0})^3 + b_1(\frac{n}{n_0})^2 Q + b_2(\frac{n}{n_0})Q^2 + ... \tag{46}$$

The pump performance curve $H(Q)$ is known. We have $p_1^{i+1} = p_s + \rho g H(Q)$, where p_s is the pressure at the suction side of the pump. Furthermore, we have $Q = A_p v_1^{i+1}$ (A_p is the cross section area of the pipe), thus the equations to be solved for p_1^{i+1} and Q is

$$p_1^{i+1} = p_s + \rho g H(Q) \quad \text{and} \quad p_1^{i+1} - \rho a \frac{Q}{A_p} = K_l, \tag{47}$$

which can be rewritten as a single equation for Q:

$$P - P_s = K_l + \rho a \frac{Q}{A_p} - P_s \tag{48}$$

$$p_s + \rho g H(Q) - \rho a \frac{Q}{A_p} = K_l \tag{49}$$

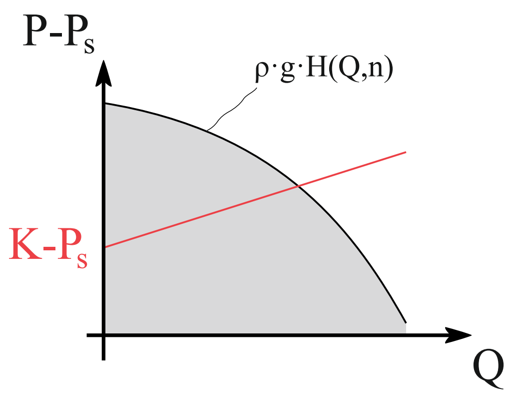


Figure 5: Graphical solution of the pump equation.

Pump revolution number update:

1, Pump starts with prescribed $n_{(t)}$ curve

e.g. frequency converter

2, Direct pump start (motor curve)

$$\Theta \varepsilon = M_m - M_{hydr} \tag{50}$$

Where Θ: the mass moment of inertia of the motor rotating part and the impeller ε: the angular acceleration and $M_m - M_{hydr}$: the difference between the motor and hydraulic torque. This equation can be rewritten:

$$\Theta 2\pi \frac{n^{i+1} - n^i}{\triangle t} = M_{m(n^i)} - \frac{P(Q^i, n^i)}{2\pi n^i} \tag{51}$$

3, Pump run-out

e.g. pump stops due to electric failure $\rightarrow M_m = 0$

$$n^{i+1} = n^i - \underbrace{\frac{\triangle t}{\Theta} \frac{1}{(2\pi)^2} \frac{P(Q^i, n^i)}{n^i}}_{\triangle n} \tag{52}$$

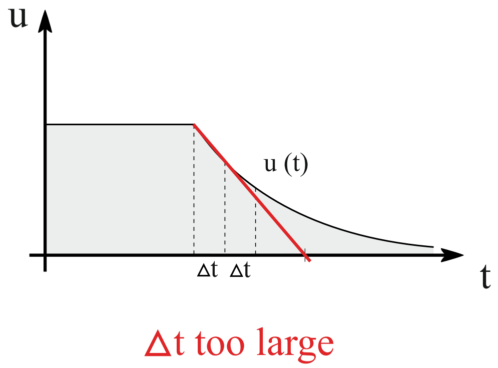


Figure 6: Pump run-out.

4 Impedance method

The impedance technique presented in this chapter allows the calculation of hydraulic eigenfrequencies of pipeline systems. The basic frequency of a pipeline of length L and sonic velocity a is $f = a/(2L)$, where $2L/a$ is the time scale of the pipe; i.e. the time needed for a pressure wave to travel to the other end of the time and come back. However, for more complicated pipelines, it is not straightforward to cope with the interaction between different pipe segments.

The impedance technique assumes *periodic flow* in the pipeline and connects the amplitude of the excitation at one end with the response amplitude on the other end of the pipe, for arbitrary excitation frequency. Thus with the help of this technique, it is possible to construct resonance diagrams of complex (tree-like or looped) pipeline systems.

4.1 Basic theory

We start from the one dimensional continuity and momentum equation. The convective terms are neglected in both equations. We denote the sum of the static pressure p_{st} and hydrostatic pressure ρgh by p

$$p = p_{st} + \rho gh,$$

thus the basic equations are:

$$\frac{\partial p}{\partial x} + \rho \frac{\partial v}{\partial t} + \frac{\rho \lambda}{2d} v\,|v| = 0, \tag{53}$$

$$\frac{1}{a^2}\frac{\partial p}{\partial t} + \rho \frac{\partial v}{\partial x} = 0. \tag{54}$$

We consider *only periodic flows*. Let the mean values of pressure and velocity be $\bar{p}$ and $\bar{v}$ respectively, and the periodic parts be denoted by p and v:

$$p = \bar{p} + p'; \quad v = \bar{v} + v'. \tag{55}$$

The mean values are defined by the time integrals:

$$\bar{p}\,(x) = \frac{1}{T}\int_0^T p(t,x)dt \quad \text{and} \quad \bar{v}(x) = \frac{1}{T}\int_0^T v(t,x)dt. \tag{56}$$

In order to substitute (55) into (53) and (54) one has to differentiate the pressure and velocity:

$$\frac{\partial p}{\partial x} = \frac{\partial \bar{p}}{\partial x} + \frac{\partial p'}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial \bar{v}}{\partial x} + \frac{\partial v'}{\partial x}. \tag{57}$$

The derivatives with respect to time are similar. By Eq. (56)

$$\frac{\partial \bar{p}}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \bar{v}}{\partial t} = 0. \tag{58}$$

The mean values are solutions of Eqs. (53), (54) because they mean the unperturbed case:

$$\frac{\partial \bar{p}}{\partial x} = -\frac{\lambda\rho}{2d}\bar{v}\,|\bar{v}| = -\frac{\lambda\rho}{2d}\bar{v}^2 \tag{59}$$

and

$$\frac{\partial \bar{v}}{\partial x} = 0. \tag{60}$$

Further, we suppose that the velocity perturbation is much smaller than the mean value: $v' \ll \bar{v}$. Then we can neglect v'^2 in the turbulent friction term:

$$\frac{\lambda\rho}{2d}v^2 \cong \frac{\lambda\rho}{2d}\left(\bar{v}^2 + 2\bar{v}v'\right). \tag{61}$$

Substituting (57) – (61) into (53) and (54) we get:

$$\frac{\partial p'}{\partial x} + \rho\frac{\partial v'}{\partial t} + R_0 v' = 0, \tag{62}$$

$$\frac{\partial p'}{\partial t} + \rho\, a^2\frac{\partial v'}{\partial x} = 0, \qquad \text{with} \tag{63}$$

$$R_0 = \frac{\rho\lambda}{d}\bar{v}. \tag{64}$$

By differentiating one of Eqs. (62) and (63) with respect to time, the other one with respect to x and subtracting the second from the first results in equations which contain only p' or v'. ($R = R_0/\rho$):

$$a^2\frac{\partial}{\partial x}(62) - \frac{\partial}{\partial t}(63): \qquad a^2\frac{\partial^2 p'}{\partial x^2} - \frac{\partial^2 p'}{\partial t^2} - R\frac{\partial p'}{\partial t} = 0, \tag{65}$$

$$-\frac{1}{\rho}\left[\frac{\partial}{\partial t}(62) - \frac{\partial}{\partial x}(63)\right]: \qquad a^2\frac{\partial^2 v'}{\partial x^2} - \frac{\partial^2 v'}{\partial t^2} - R\frac{\partial v'}{\partial t} = 0. \tag{66}$$

The similar equations are both solved by Fourier's separation method. This will result in complex pressure and velocity perturbations: Naturally, only the real part has physical meaning. $\left(\imath = \sqrt{-1}\right)$

$$p'(x,t) = e^{\imath\omega t}\left(Ae^{\gamma x} + Be^{-\gamma x}\right). \tag{67}$$

Putting this into (63) and integrating over the pipe length x we get

$$v'(x,t) = \frac{\omega}{\imath\rho a^2\gamma}e^{\imath\omega t}\left(Ae^{\gamma x} - Be^{-\gamma x}\right). \tag{68}$$

In the above equations ω is the frequency of excitation, A and B will be determined by the boundary values, γ is called *propagation constant.* It is defined by

$$\gamma^2 = -\frac{\omega^2}{a^2} + \imath\frac{R\omega}{a^2}.$$

As a new parameter the hydraulic impedance is introduced, it is the ratio of pressure perturbation p' and velocity perturbation v':

$$Z(x) = \frac{p'}{v'} = \frac{\imath\rho a^2\gamma}{\omega}\cdot\frac{Ae^{\gamma x} + Be^{-\gamma x}}{Ae^{\gamma x} - Be^{-\gamma x}} \tag{69}$$

The first brake is called characteristic impedance Z_c:

$$Z_c = -\frac{\imath\rho a^2\gamma}{\omega} \tag{70}$$

The hydraulic impedance $Z(x)$ depends only on the space coordinate. At the upstream end of the hydraulic element, $Z_u = Z(x=0)$. At the downstream end $Z_d = Z(x=L)$.

$$Z_u = Z(0) = \frac{p'(0,t)}{v'(0,t)} \tag{71}$$

$$Z_d = Z(L) = \frac{p'(L,t)}{v'(L,t)} \tag{72}$$

The coefficients A and B can be expressed with the pressure and velocity perturbation at the upstream end of the hydraulic element. Putting $x = 0$ into (67) and (68) one has

$$p'(0,t) = e^{\imath\omega t}(A+B) = P_u, \tag{73}$$

$$v'(0,t) = -\frac{e^{\imath\omega t}}{Z_c}(A-B) = V_u. \tag{74}$$

These give for A and B

$$A = \frac{1}{2e^{\imath\omega t}}\left(P_u - Z_c V_u\right), \tag{75}$$

$$B = \frac{1}{2e^{\imath\omega t}}(P_u + Z_c V_u). \tag{76}$$

The perturbations over the length of the element are now:

$$p'(x,t) = P_u \cosh\gamma x - Z_c V_u \sinh\gamma x, \tag{77}$$

$$v'(x,t) = -\frac{P_u}{Z_c}\sinh\gamma x + V_u\cosh\gamma x. \tag{78}$$

Using these equations we find a relation between the upstream (P_u, V_u) and downstream values (P_d, V_d) of perturbations:

$$P_d = p'(L,t) = P_u\cosh\gamma L - Z_c V_u \sinh\gamma L, \tag{79}$$

$$V_d = v'(L,t) = -\frac{P_u}{Z_c}\sinh\gamma L + V_u\cosh\gamma L. \tag{80}$$

Using (69) – (72) and Eqs. (77), (78) the hydraulic impedance $Z(x)$ can be written as:

$$Z(x) = \frac{Z_u - Z_c\tanh\gamma x}{1 - \frac{Z_u}{Z_c}\tanh\gamma x}.$$

With vector notation and putting $x = L$ one has:

$$\begin{pmatrix} P_d \\ V_d \end{pmatrix} = \begin{pmatrix} \cosh\gamma L & -Z_c\sinh\gamma L \\ -\frac{\sinh\gamma L}{Z_c} & \cosh\gamma L \end{pmatrix} \begin{pmatrix} P_u \\ V_u \end{pmatrix}.$$

Or:

$$\begin{pmatrix} P_u \\ V_u \end{pmatrix} = \begin{pmatrix} \cosh\gamma L & Z_c \sinh\gamma L \\ \frac{\sinh\gamma L}{Z_c} & \cosh\gamma L \end{pmatrix} \begin{pmatrix} P_d \\ V_d \end{pmatrix}.$$

The matrix is called *impedance matrix*. The resulting impedance matrix of hydraulic elements connected in series is the product of the impedance matrices of the individual elements. The following expressions connect the impedances at the upstream and downstream ends of the element:

$$Z_d = \frac{Z_u - Z_c \tanh\gamma L}{1 - \frac{Z_u}{Z_c}\tanh\gamma L}.$$

$$Z_u = \frac{Z_d + Z_c \tanh\gamma L}{1 + \frac{Z_d}{Z_c}\tanh\gamma L}.$$

4.2 Boundary conditions

Some simple cases are studied where either the upstream or the downstream impedance (Z_u or Z_d)can be found easily. The pressure perturbation is zero if the *downstream pressure has a fixed constant value* (open end to the atmosphere or a liquid tank with constant surface)

$$Z_d = \frac{p'(L,t)}{v'(L,t)} = 0$$

Closed end of a pipe, the velocity perturbation is zero, thus

$$Z_d = \infty$$

Dividing or combining pipes (or other hydraulic elements) results in a boundary condition where the pressure perturbation is common for all connected elements and continuity is fulfilled. For elements k being connected ($k = 1, 2, \ldots, K$):

$$p'_1(L_i,t) = p'_2(L_2,t) = \ldots = p'_k(L_K,t) \tag{81}$$

Supposing constant liquid density in elements having cross sections A_k gives:

$$\sum_{k=1}^{K} A_k \left(\bar{v}_k + v'_k\right) = 0.$$

Continuity must be fulfilled for the mean velocities $\bar{v}$ too,

$$\sum_{k=1}^{K} A_k v'_k = 0. \tag{82}$$

From (81) and (82)

$$\sum_{k=1}^{K} \frac{A_k}{Z_k} = 0.$$

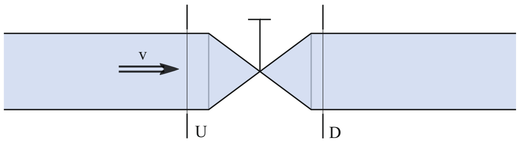


Figure 7: Throttle valve.

The fact that the velocity is proportional to the square root of pressure difference in turbulent flow is used to formulate the impedance of a *throttle valve* at the downstream end of a pipe:

$$\bar{v} + v' = \mu\sqrt{\frac{2}{\rho}\left(\Delta\bar{p} + \Delta p'\right)}.$$

If the pressure outside the valve is constant we can write $\Delta\bar{p} = \bar{p}$ and $\Delta p' = p'$ thus:

$$\bar{v} + v' = \mu\sqrt{\frac{2}{\rho}\left(\bar{p} + p'\right)}. \tag{83}$$

Considering that in many cases $p' \ll \bar{p}$ we can write

$$\bar{p} + p' \cong \bar{p}\left(1 + \frac{p'}{2\bar{p}}\right)^2,$$

further, for steady flow

$$\bar{v} = \mu\sqrt{\frac{2}{\rho}\bar{p}}.$$

After substituting these into (83) we get:

$$v' = \bar{v}\frac{p'}{2\bar{p}}.$$

Thus the downstream impedance is

$$Z_d = \frac{2\bar{p}}{\bar{v}}.$$

This is a real number meaning that there is no phase shift between the velocity and pressure variation in this point. As the last step we formulate the *excitation* as a boundary condition. Both velocity and pressure excitation can be handled in a similar manner. Let's see the velocity excitation. If the real velocity excitation is a sinusoidal vibration with angular frequency ω and amplitude A_0:

$$\Re\left(v'\right) = A_0 \sin\omega t$$

then the complex form of v' is:

$$v' = A_0 e^{\imath\left(\omega t - \frac{\pi}{2}\right)}.$$

Really:

$$v' = A_0 e^{\imath\left(\omega t - \frac{\pi}{2}\right)} = A_0(\cos\omega t + i\sin\omega t)(-i) = -iA_0\cos\omega t + A_0\sin\omega t$$

As a further step we define the impedance of a turbopump. In the figure below we see the characteristics of the pump and the system intersecting each other in the working point. The pump characteristics may be approximated by its tangent in the working point.

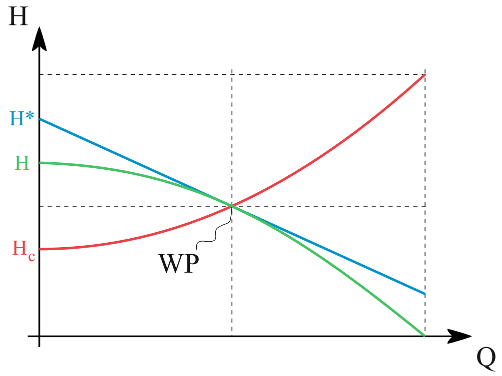


Figure 8: Pump characteristics.

This tangent has the equation:

$$H^* = A - BQ = A - \frac{dH}{dQ}|Q = Q_W PQ \tag{84}$$

The slope of the characteristic curve in the working point can be determined graphically or numerically. Once this has been done the head and flow rate can be expressed by the pressure and velocity.

$$p_d - p_u = \overline{p}_d + p_d' - (\overline{p}_u + p_u') = \rho g H^* = \rho g(A - BQ) \tag{85}$$

$$Q = \frac{D^2\pi}{4} v_u = \frac{D^2\pi}{4}(\overline{v}_u + v_u') \tag{86}$$

Finally we suppose that the velocity doesnt change spatially through the pump: $v_d = v_u$ which holds also for the perturbed values: $v_d' = v_u'$.

We obtain:

$$\underline{\overline{p}_d - \overline{p}_u} + p_d' - p_u' = \underline{\rho g}(\underline{A - B\frac{D^2\pi}{4}}(\underline{\overline{v}_u} + v_u') \tag{87}$$

The terms underlined are equal as they represent the steady state working point. Thus; if we now denote the perturbances by upper case letters as before, we have:

$$P_d - P_u = p_d' - p_u' = -[\rho g B\frac{D^2\pi}{4}]v_u' = -Kv_u' = -KV_u \tag{88}$$

In matrix form:

$$\begin{pmatrix} P_d \\ V_d \end{pmatrix} = \begin{pmatrix} 1 & -K \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P_u \\ V_u \end{pmatrix}.$$

Or the connection of impedances of a pump is:

$$Z_d = Z_u - K \tag{89}$$

Surge tank:

The pressure above the free surface in the surge tank is not perturbed; it is the constant atmospheric pressure p_0. The steady state liquid height is H_0, it is perturbed, the varying water level h is measured from H_0. The cross sectional area A of the surge tank is constant along its length. The velocity perturbation throughout the tank is v, it is equal to the derivative of h, thus $h' = -\int_0^1 v'dt$.

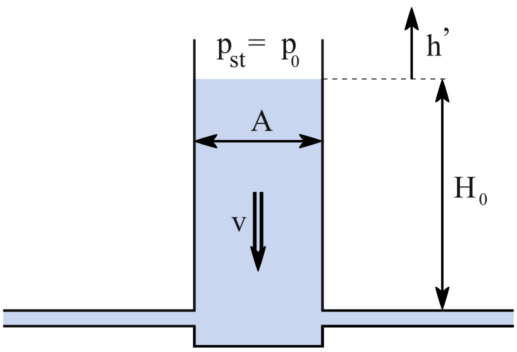


Figure 9: Surge tank.

The actual water depth in the tank is:

$$h = H_0 + h' = H_0 - \int_0^1 v'dt \tag{90}$$

In the introduction we have defined the pressure as $p = p_st + \rho gh$, thus:

$$p(x,t) = p_st + \rho g(H_0 + h') = p_st + \rho g H_0 + \rho g h' = \overline{p} + \rho g h' \tag{91}$$

We obtain from here the following:

$$p' = -\rho g \int_0^1 v'dt \tag{92}$$

As $p' = Pe^{i\omega t}$, we have $\frac{dp'}{dt} = Pi\omega e^{i\omega t}$ and from the above integral $\frac{dp'}{dt} = -\rho g v' = -\rho g V e^{i\omega t}$. By combinig these $Pi\omega = -\rho g V$ or $P = i\frac{\rho g}{\omega}V$. From here the impedance is:

$$Z = i\frac{\rho g}{\omega} \tag{93}$$

This result shows that there is a 90° phase shift between the pressure and velocity perturbation at tank bottom.

Example:

A surge tank of $A_1 = 10m^2$ cross section feeds a frictionless pipe of length L and cross section $A = 1m^2$ with water. At the downstream end of the pipe another surge tank of cross section $A_2 = 20m^2$ is located. Lets find the resonance frequencies of this system for varying pipe lengths. The impedance of the tank bottom is $Z_1 = i\frac{\rho g}{\omega}$. This point is attached to the upstream end of the pipe. At this junction the flow directions are equal, thus $\frac{A_1}{Z_1} = \frac{A}{Z_u}$, $Z_u = Z_1\frac{A}{A_1} = i\frac{A}{A_1}\frac{\rho g}{\omega}$. The downstream impedance of the frictionless pipe can now be computed. The propagation factor for frictionless flow is $\gamma = i\frac{\omega}{a}$ and the characteristic impedance is $Z_c = -i\frac{\rho a^2}{\omega}i\frac{\omega}{a} = \rho a$.

With these:

$$Z_d = \frac{Z_u - Z_c \tanh(\gamma L)}{1 - \frac{Z_u}{Z_c}\tanh(\gamma L)} \tag{94}$$

$$Z_d = \frac{i\frac{A}{A_1}\frac{\rho g}{\omega} - \rho a \tanh(i\frac{\omega}{a}L)}{1 - i\frac{A}{A_1}\frac{\rho g}{\omega}\frac{1}{\rho a}\tanh(i\frac{\omega}{a}L)} \tag{95}$$

$$Z_d = i\rho a\frac{\frac{A}{A_1}\frac{g}{\omega a} - \tanh(\frac{\omega}{a}L)}{1 + \frac{A}{A_1}\frac{g}{\omega a}\tanh(\frac{\omega}{a}L)} \tag{96}$$

We used the $\tanh(i\gamma) = i\tan(\gamma)$ identity of complex arithmetic. The impedance of the bottom of the second surge tank is $Z = Z_d\frac{A_2}{A}$ because of the opposite flow directions in the pipe and in the tank.

We obtain:

$$|Z| = \rho a\frac{A_2}{A}\left|\frac{\frac{A}{A_1}\frac{g}{\omega a} - \tanh(\frac{\omega}{a}L)}{1 + \frac{A}{A_1}\frac{g}{\omega a}\tanh(\frac{\omega}{a}L)}\right| = f(\omega, L) \tag{97}$$

5 Unsteady 1D open-surface flow in prismatic channel

5.1 Introduction - steady-state flow in a pritzmatic channel

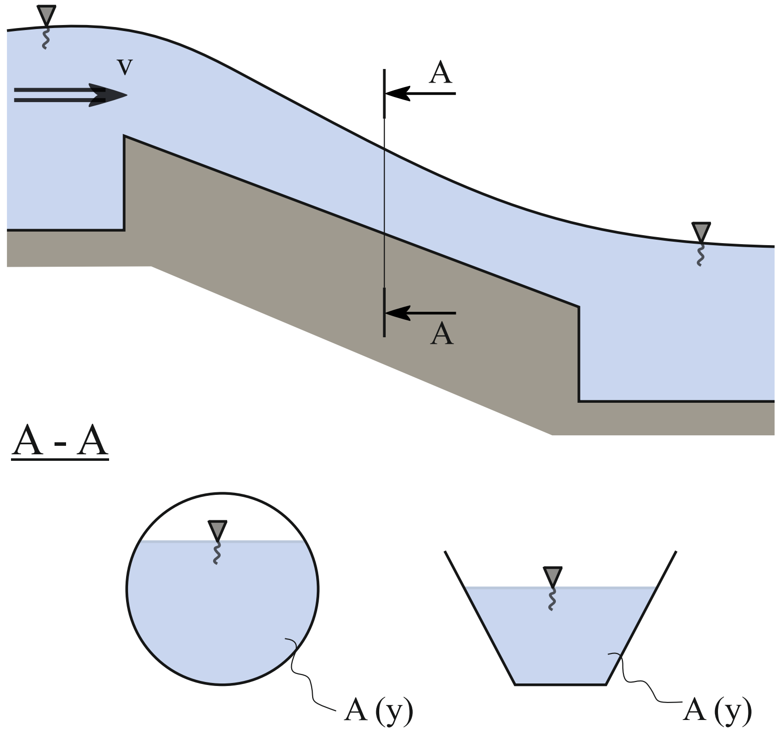


Figure 10: Open surface flow (schematic figure).

Typical applications:

- storm water system
- waste water

Upon applying Bernoulli's equation, we have

$$\frac{d}{dx}\left(y(x)+\frac{v(x)^2}{2g}+z+h'(x)\right)=0, \tag{98}$$

where the friction loss is $\frac{d}{dx}\left(\frac{\Delta p}{\rho g}\right)=\frac{v^2}{c^2 R_h}$ with $R_h=\frac{A}{p}$ the ration of the area and the perimeter (see Figure **??**) and $c=R_h^{1/6}\frac{1}{n}$ where n is the so-called *Manning's constant* (similar to λ). As the velocity is related to the flow rate as $v(x)=Q/A(x)$, we have

$$\frac{d}{dx}\left(\frac{v(x)^2}{2g}\right)=\frac{1}{2g}\frac{d}{dx}\left(\frac{Q^2}{A(x)^2}\right)=-2\frac{Q^2}{2g}\frac{1}{A(x)^3}\frac{dA(x)}{dx}. \tag{99}$$

5.2 Continuity equation and equation motion for open-surface flow

The usual form of continuity equation: $\frac{\partial \rho}{\delta t} + \frac{\partial}{\delta x}(\rho v) = 0$ is not suitable, because $\rho = const$ in our case; and the above form assumes constant flow-through area (which is not true for OCF).

5.2.1 Continuity equation

Using Taylor expansion in space around the inlet, the volume leaving at the end in dt time is:

$$\rho_2 A_2 v_2 dt = \left(\rho_1 A_1 v_1 + \frac{\partial}{\partial x}(\rho A v) dx \right) dt \tag{100}$$

The mass of fluid in the stream tube is

$$m(t + dt) = m(t) + \frac{\partial m}{\partial t} dt = m(1) + \frac{\partial}{\partial t}(\rho A dx) dt \tag{101}$$

Putting these together we finally get

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A v) = 0 \tag{102}$$

Notice that

- if $A = const$ then get back the usual form: $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$ and
- if $\rho = const$ then $\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$ (the volumetric flow rate is $Q = Av$).

5.2.2 Equation of motion

From Newton's II. law we know that

$$\frac{dF}{dt} = F_{pressure} + F_{gravity} + F_{friction} \tag{103}$$

The general form is

$$\frac{\partial \rho v A}{\partial t} + \frac{\partial (v \rho v A)}{\partial x} = -A \left(\frac{\partial p}{\partial x} - \rho g S_0 + \rho g S_f \right) \tag{104}$$

where $S_0 = -dz/dx$ is the bed slope and $S_f = \frac{f}{2Dg} v^2$ is the friction loss.

5.2.3 General vector form for open-surface flow

For a general compressible fluid, the continuity equation reads

$$\frac{\partial \rho A}{\partial t} + \frac{\partial \rho A v}{\partial x} = 0. \tag{105}$$

and the momentum equation is

$$\frac{\partial \rho A v}{\partial t} + \frac{\partial\, \rho A v^2}{\partial x} = -A\left[\frac{\partial p}{\partial x} + \rho g \frac{\partial z}{\partial x} + \frac{\rho}{2}\frac{\lambda}{d} v|v|\right]. \tag{106}$$

If the density is considered to be constant, the continuity equation becomes

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \tag{107}$$

while the equation of motion is

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -A\left[\frac{1}{\rho}\frac{\partial(\rho g y)}{\partial x} + g\frac{\partial z}{\partial x} + \frac{\lambda}{2d}v|v|\right] = -gA\frac{\partial y}{\partial x} + gA(S_0 - S_f). \tag{108}$$

Now rewrite the first temr on the RHS as

$$A\frac{\partial y}{\partial x} = \frac{\partial A y}{\partial x} - y\frac{\partial A}{\partial x} = \frac{\partial A y}{\partial x} - y\frac{\partial A}{\partial y}\frac{\partial y}{\partial x} \tag{109}$$

and we arrive at

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + gAy\right) = gyB\frac{\partial y}{\partial x} + gA(S_0 - S_f). \tag{110}$$

5.3 Shallow water waves

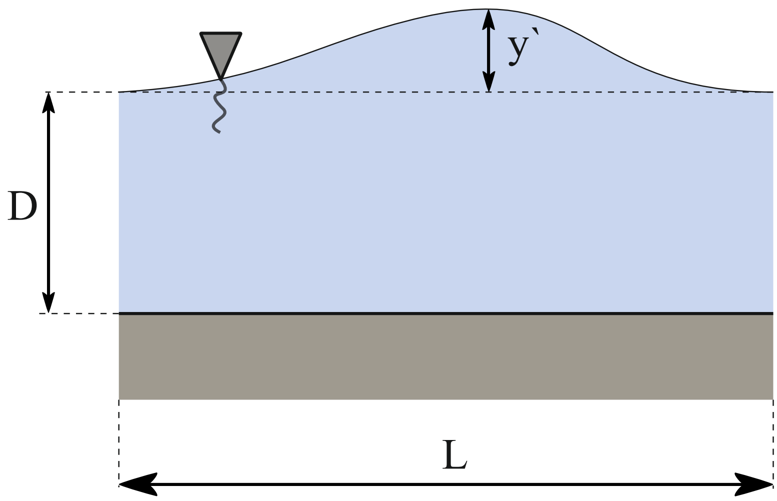


Figure 11: Shallow water wave.

For shallow water waves: $y' << D$ and the mean flow velocity $v << \frac{L}{T}$, where D is the average depth, L is the wave length and T is the characteristic time. Assume that $A = By$ and $y = D + y'$. Continuity:

$$\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x} = -y\frac{\partial v}{\partial x} \tag{111}$$

$$\frac{\partial y'}{\partial t} + \underbrace{v\frac{\partial y'}{\partial x}}_{\approx 0} = -(D + y')\frac{\partial v}{\partial x} \tag{112}$$

Linearization:

$$\frac{\partial y'}{\partial t} = -D\frac{\partial v}{\partial x} \tag{113}$$

Momentum equation: (Source terms are zero.)

$$S_0 = S_f = 0 \rightarrow \frac{\partial v}{\partial t} = -g\frac{\partial y'}{\partial x} \tag{114}$$

From these, we obtain:

$$\frac{\partial^2 y'}{\partial t^2} = -D\frac{\partial}{\partial x}(-g\frac{\partial y'}{\partial x}) \tag{115}$$

Wave equation:

$$\frac{\partial^2 y'}{\partial t^2} - Dy\frac{\partial^2 y'}{\partial x^2} = 0 \tag{116}$$

Where, $Dy = c^2$ and c is the wave celerity. So the shallow water wave celerity is $c = \sqrt{gD}$.

5.4 The Saint-Venant equations

On typical way of writing the above equations is to introduce the free-surface width as $B(y) = \frac{\mathrm{d}A(y)}{\mathrm{d}y}$, hence, for example

$$\frac{\partial A}{\partial t} = \frac{dA}{dy}\frac{\partial y}{\partial t} := B(y)\frac{\partial y}{\partial t}. \tag{117}$$

The continuity equation can be written as

$$0 = \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = B\left(\frac{\partial y}{\partial t} + \frac{A}{B}\frac{\partial v}{\partial x} + v\frac{\partial y}{\partial x}\right) \tag{118}$$

The equation of motion is

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}+gAy\right)=gyB\frac{\partial y}{\partial x}+Ag\left(S_0-S_f\right) \tag{119}$$

$$A\frac{\partial v}{\partial t}+Bv\frac{\partial y}{\partial t}+A2v\frac{\partial v}{\partial x}+v^2B\frac{\partial y}{\partial x}+gA\frac{\partial y}{\partial x}+gyB\frac{\partial y}{\partial x}=gyB\frac{\partial y}{\partial x}+Ag\left(S_0-S_f\right) \tag{120}$$

$$A\frac{\partial v}{\partial t}+Bv\left(\frac{\partial y}{\partial t}+\frac{A}{B}\frac{\partial v}{\partial x}+v\frac{\partial y}{\partial x}\right)+Av\frac{\partial v}{\partial x}+gA\frac{\partial y}{\partial x}+gyB\frac{\partial y}{\partial x}=gyB\frac{\partial y}{\partial x}+Ag\left(S_0-S_f\right) \tag{121}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial y}{\partial x}=g\left(S_0-S_f\right) \tag{122}$$

The famous Saint-Venant equations are

$$\frac{\partial y}{\partial t}+\frac{A}{B}\frac{\partial v}{\partial x}+v\frac{\partial y}{\partial x}+\frac{v}{B}\left.\frac{\partial A}{\partial x}\right|_{y=const.}=0 \tag{123}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial y}{\partial x}=g\left(S_0-S_f\right) \tag{124}$$

where $S_0=-dz/dx$ and $S_f=\frac{\lambda}{2gd}v|v|=\frac{n^2}{R_h^{4/3}}v|v|$.

5.4.1 Steady-state solutions of Saint-Venant equations

Steady-state solution: $\frac{\partial \clubsuit}{\partial t}=0$

Thus, the continuity equation:

$$\frac{\partial Q}{\partial x}=0 \tag{125}$$

Q is constant along the channel.

The momentum equation:

$$\frac{dy}{dx}(1-\frac{BQ^2}{A^3y})=S_0-S_f \tag{126}$$

The cross section of the channel can be seen on 12.

From this:

$$\frac{Q^2}{A^2y\frac{A}{B}}=\frac{v^2}{c^2}=Fr^2 \tag{127}$$

$$\frac{dy}{dx}=\frac{S_0-S_f}{1-Fr^2} \tag{128}$$

S_0 is constant, $S_f(v)$ and $Fr(v,c)$.

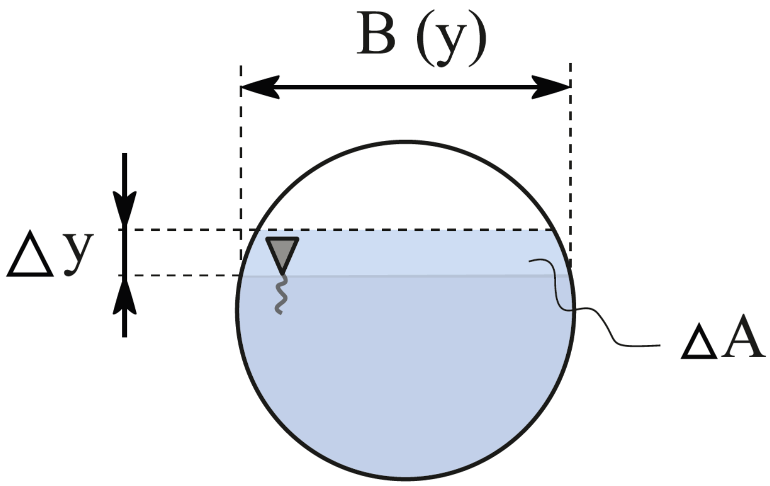


Figure 12: Channel cross section.

5.4.2 Hyperbolicity of the Saint-Venant equations

Matrix form:

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ Q \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 1 \\ c^2 - v^2 & 2v \end{pmatrix}}_{\underline{\underline{J}}} \frac{\partial}{\partial x}\begin{pmatrix} A \\ Q \end{pmatrix} = \underline{S} \tag{129}$$

As a consequence:

$$(c^2 - v^2)A + 2vQ = Agy - v^2A + 2vAv = Agy + \frac{v^2A^2}{A} = Agy + \frac{Q^2}{A} \tag{130}$$

The vectorial form of the Saint-Venant equations is:

$$\frac{\partial}{\partial t}\underline{U} + \underline{\underline{A}}\frac{\partial}{\partial x}\underline{U} = \underline{U} \tag{131}$$

We solve it according to the followings:

$$\underline{\underline{A}} \quad \underline{x} = \underline{B} \tag{132}$$

$$\underline{x} = \underline{\underline{T}} \quad \underline{y} \tag{133}$$

Where, $\underline{\underline{T}}$ is the transverse matrix.

$$\underline{\underline{AT}} \quad \underline{y} = \underline{B} \tag{134}$$

$$\underline{\underline{T^{-1}AT}} \quad \underline{y} = \underline{\underline{T^{-1}}} \quad \underline{B} \tag{135}$$

We applied linear transformation to use a new basis determined by the eigenvalues, and thus have a diagonal matrix ($\underline{\underline{T^{-1}AT}}$) instead of $\underline{\underline{A}}$.

We apply this method for our equation. First, we need the eigenvalues and eigenvectors of the matrix $\underline{\underline{J}}$. The eigenvalues are obtained by calculating the determinant of the following matrix:

$$\begin{pmatrix} 0 & 1 \\ c^2 - v^2 & 2v \end{pmatrix} \tag{136}$$

$$\lambda_{1,2} = \frac{1}{2}(2v \pm \sqrt{4v^2 - 4v^2 + 4c^2}) = v \pm c \tag{137}$$

The eigenvalues are $\lambda_1 = v - c$ and $\lambda_2 = v + c$, thus the eigenvectors are $v_1 = (\frac{1}{v-c}, 1)^T$ and $v_2 = (\frac{1}{v+c}, 1)^T$.

The vectorial form of the Saint-Vernant equations with the new basis:

$$\frac{\partial}{\partial t}\underline{\underline{T}}\underline{\tilde{U}} + \underline{\underline{A}}\frac{\partial}{\partial x}\underline{\underline{T}}\underline{\tilde{U}} = \underline{S} \tag{138}$$

$$\underline{\underline{T}}\frac{\partial \underline{\tilde{U}}}{\partial t} + \underline{\underline{AT}}\frac{\partial}{\partial x}\underline{\tilde{U}} = -\underline{\tilde{U}}\frac{\partial T}{\partial x} - \underline{\underline{A}}\underline{\tilde{U}}\frac{\partial \underline{\underline{T}}}{\partial x} + \underline{S} \tag{139}$$

$$\frac{\partial \underline{\tilde{U}}}{\partial t} + \begin{pmatrix} v + c & 0 \\ 0 & v - c \end{pmatrix} \frac{\partial}{\partial x}\underline{\tilde{U}} = RHS \tag{140}$$

This is of the type $U_t + aU_x = 0 \rightarrow$ hyperbolic Partial Differential Equation. Not practical for numerical solution due to the right hand side term.

5.5 Open-surface channel flow and gas dynamics

There are some similar phenomena in gas dynamics:

Gas dynamics	$\rightarrow$	**Open-surface flow**
$M = \frac{v}{a}$		$F_r = \frac{v}{c}$
M: Mach number		F_r: Froude number
$M < 1$ subsonic		$F_r < 1$ subcritical
$M > 1$ supersonic		$F_r > 1$ supercritical
shock wave		hydraulic jump

5.6 Method of characteristics for open-surface flow

5.6.1 MOC formulation for rectangular channel

The Saint-Venant equation can be re-written in terms of wave celerity $c = \sqrt{gy}$ instead of water depth. First, we compute

$$\frac{\partial y}{\partial t} = \frac{1}{g}\frac{\partial c^2}{\partial t} = \frac{2c}{g}\frac{\partial c}{\partial t} \quad \text{and} \quad \frac{\partial y}{\partial x} = \frac{1}{g}\frac{\partial c^2}{\partial x} = \frac{2c}{g}\frac{\partial c}{\partial x}. \tag{141}$$

we have now

$$\frac{2c}{g}\frac{\partial c}{\partial t} + \frac{A}{B}\frac{\partial v}{\partial x} + v\frac{2c}{g}\frac{\partial c}{\partial x} = 0 \tag{142}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + 2c\frac{\partial c}{\partial x} = g\left(S_0 - S_f\right) \tag{143}$$

Upon adding the two equations, i.e. EoM + $K \times$ Continuity:

$$\left(\frac{\partial v}{\partial t} + K\frac{2c}{g}\frac{\partial c}{\partial t}\right) + v\left(\frac{\partial v}{\partial x} + K\frac{2c}{g}\frac{\partial c}{\partial x}\right) + \left(2c\frac{\partial c}{\partial x} + Ky\frac{\partial v}{\partial x}\right) = 0. \tag{144}$$

Upon choosing $K = \pm\frac{g}{c}$, we have:

$$\left(\frac{\partial v}{\partial t} \pm 2\frac{\partial c}{\partial t}\right) + v\left(\frac{\partial v}{\partial x} \pm 2\frac{\partial c}{\partial x}\right) + c\left(2\frac{\partial c}{\partial x} \pm \frac{\partial v}{\partial x}\right) = g\left(S_0 - S_f\right) \quad \text{or, upon rearranging,} \tag{145}$$

$$\frac{\partial}{\partial t}\left(v \pm 2c\right) + \left(v \pm c\right)\frac{\partial}{\partial x}\left(v \pm 2c\right) = g\left(S_0 - S_f\right). \tag{146}$$

5.6.2 Numerical technique for the internal points

In the case of open-surface flows, there are several differences between the slightly compressible case (pressurized pipeline systems):

- $c = \sqrt{gy}$ not constant
- $v << c$ can happen (e.g:$y = 1\frac{m}{s} \to c \approx 3\frac{m}{s}$)
- even $v > c$ can happen!

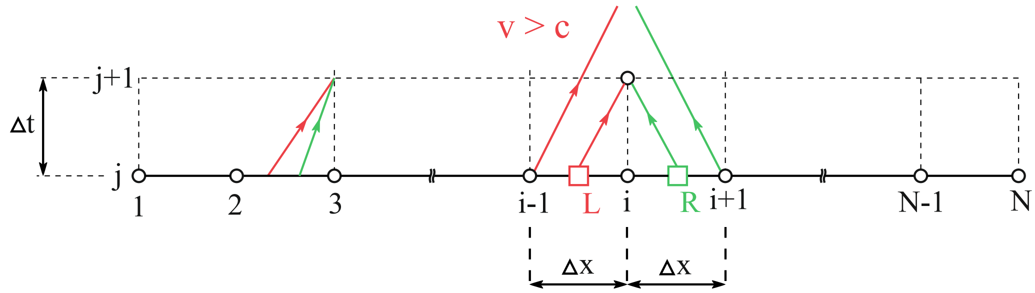


Figure 13: Characteristic grid.

Timestep selection

We have to choose appropriate timestep that meets the CFL criteria, thus we calculate the time needed to travel Δx distance with $v \pm c$ velocity, for each grid point: $\Delta t_i^+ = \frac{\Delta x}{|v_i + c_i|}$ and $\Delta t_i^- = \frac{\Delta x}{|v_i - c_i|}$. The final timestep for all grid points is than $\delta t = \min_i \left(\Delta t_i^+, \Delta t_i^-\right)$.

To avoid crossing cells(see 14), the Courant-friedrichs-lewy number can be applied.

$$\Delta t = CFL \min_i \left(\frac{\Delta x}{|v_i| + c_i}\right) \tag{147}$$

Where CFL is around 0.9, but minimum 0.5.

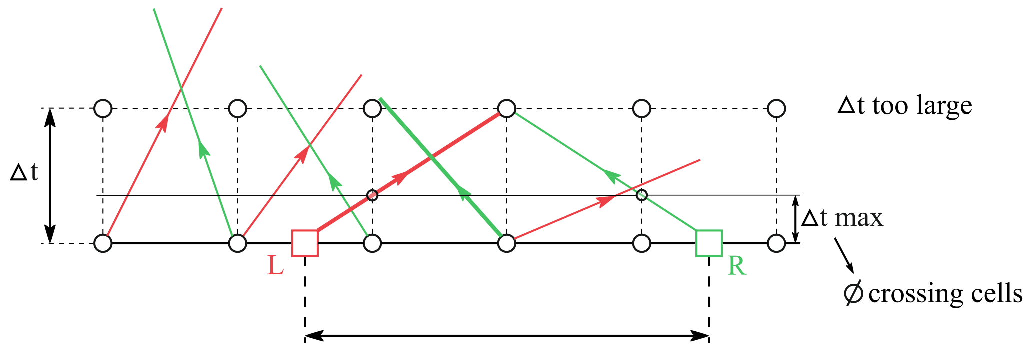


Figure 14: Crossing cells.

Internal points

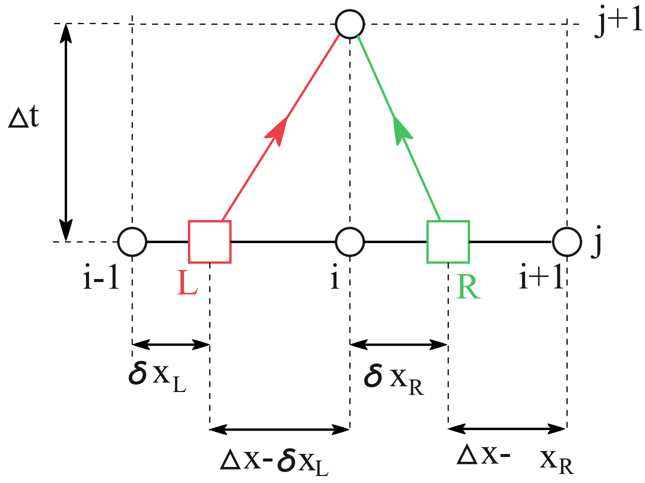


Figure 15: Update of internal points.

One has to find the location of that L(eft) and R(ight) point, from which the characteristic lines hit the required point on the new time level, see Figure 15. We use linear interpolation (see Figure **??**), e.g. for the L point:

$$v_L(\delta x) = v_{i-1}\left(1 - \frac{\delta x_L}{\Delta x}\right) + v_i \frac{\delta x_L}{\Delta x} \tag{148}$$

$$c_L(\delta x) = c_{i-1}\left(1 - \frac{\delta x_L}{\Delta x}\right) + c_i \frac{\delta x_L}{\Delta x} \tag{149}$$

and the condition $\frac{\Delta t}{\Delta x - \delta x_L} = \frac{1}{v_L(\delta x) + c_L(\delta x)}$, which gives one single interpolation for δx_L.

$$\delta x_L = \Delta x \frac{-c_{i-1} + v_{i-1} + \frac{\Delta x}{\Delta t}}{c_i - c_{i-1} + v_i - v_{i-1} + \frac{\Delta x}{\Delta t}} \tag{150}$$

If $\delta x_L < 0$, error occurs in Δt.
If $0 < \delta x_L < \Delta x$, appropriate δx_L.
If $\Delta x < \delta x_L < 2\Delta x$, $-v_L < -c_L$ supercritical backflow occurs.

Recompute left point:

$$v_L(\delta x) = v_i\left(1 - \frac{\delta x_L^-}{\Delta x}\right) + v_{i+1} \frac{\delta x_L^-}{\Delta x} \tag{151}$$

$$c_L(\delta x) = c_i\left(1 - \frac{\delta x_L^-}{\Delta x}\right) + c_{i+1} \frac{\delta x_L^-}{\Delta x} \tag{152}$$

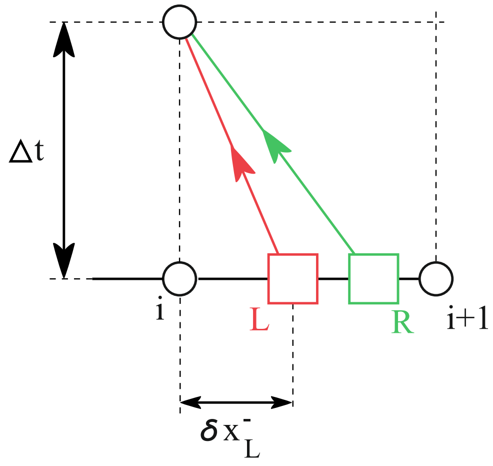


Figure 16: Recompute left point.

$$\frac{1}{v_L + c_L} = -\left(\frac{\delta x_L^-}{\Delta t}\right)^{-1} \tag{153}$$

$$\delta x_L^- = \Delta x \frac{c_i + v_i}{c_i - c_{i+1} + v_i - v_{i+1} + \frac{\Delta x}{\Delta t}} \tag{154}$$

If $\delta x_l > 2\Delta x$, error in Δt.

The R point location is found in a similar way.

$$v_R(\delta x) = v_i \left(1 - \frac{\delta x_R}{\Delta x}\right) + v_{i+1}\frac{\delta x_R}{\Delta x} \tag{155}$$

$$c_R(\delta x) = c_i \left(1 - \frac{\delta x_R}{\Delta x}\right) + c_{i+1}\frac{\delta x_R}{\Delta x} \tag{156}$$

and the condition $-\frac{\Delta t}{\delta x_R} = \frac{1}{v_R(\delta x) + c_R(\delta x)}$, which gives one single interpolation for δx_R.

$$\delta x_R = \Delta x \frac{c_i - v_i}{c_i - c_{i+1} - v_i + v_{i+1} + \frac{\Delta x}{\Delta t}} \tag{157}$$

If $\delta x_R > \Delta x$, error occurs in Δt.

If $0 < \delta x_R < \Delta x$, appropriate δx_L.

If $-\Delta x < \delta x_R < 0$, supercritical backflow occurs.

Recompute right point:

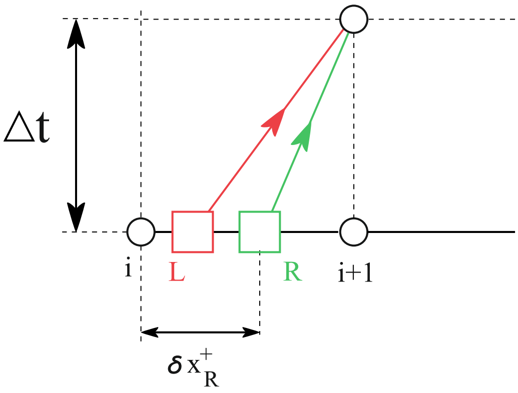


Figure 17: Recompute right point.

$$v_R(\delta x) = v_{i-1}\left(1 - \frac{\delta x_R^+}{\Delta x}\right) + v_i \frac{\delta x_R^+}{\Delta x} \tag{158}$$

$$c_R(\delta x) = c_{i-1}\left(1 - \frac{\delta x_R^+}{\Delta x}\right) + c_i \frac{\delta x_R^+}{\Delta x} \tag{159}$$

$$\frac{1}{v_R - c_r} = \frac{\Delta t}{\Delta x - \delta x_R} \tag{160}$$

$$\delta x_R^+ = \Delta x \frac{c_{i-1} - v_{i-1} + \frac{\Delta x}{\Delta t}}{-c_i + c_{i-1} + v_i - v_{i-1} + \frac{\Delta x}{\Delta t}} \tag{161}$$

If $\delta x_R < -\Delta x$, error in Δt.

We use the standard MOC method to update the water depth and the velocity: $\alpha_L = v_{i-1}^{j+1} + 2c_{i-1}^{j+1}$ and $\beta_L = v_{i+1}^{j+1} - 2c_{i+1}^{j+1}$.

$$\frac{D^+\alpha}{Dt^+} = g(S_0 - S_f) \rightarrow \frac{\alpha_i^{j+1} - \alpha_L}{\Delta t} = g(S_0 - S_f(v_L)) \tag{162}$$

$$\frac{D^-\beta}{Dt^-} = g(S_0 - S_f) \rightarrow \frac{\beta_i^{j+1} - \beta_R}{\Delta t} = g(S_0 - S_f(v_R)) \tag{163}$$

Finally:

$$v_i^{j+1} = \frac{\alpha_i^{j+1} + \beta_i^{j+1}}{2} \tag{164}$$

$$c_i^{j+1} = \frac{\alpha_i^{j+1} - \beta_i^{j+1}}{4} \tag{165}$$

$$y_i^{j+1} = \frac{(c_i^{j+1})^2}{g} \tag{166}$$

Boundary conditions

Left boundary

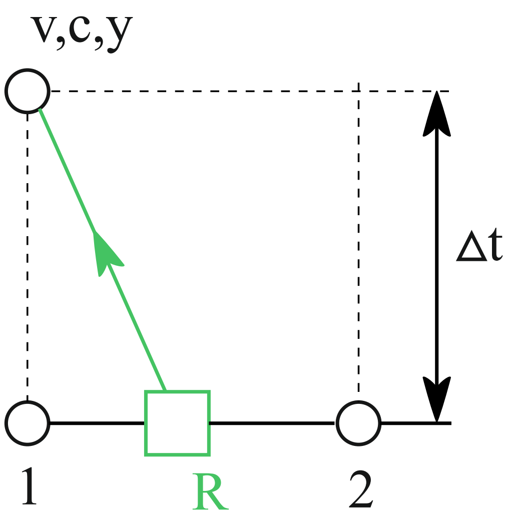


Figure 18: Left boundary.

The first used equation:

$$v - 2c = \beta_R + \Delta t g(S_0 - S_f) = K_R \tag{167}$$

Wall: $v = 0$

The second used equation is:

$$c = -\frac{K_R}{2} \rightarrow y = \frac{c^2}{g} \tag{168}$$

Depth: $y = y_1$

The second used equation is:

$$c = \sqrt{yg} \rightarrow v = K_R + 2c \tag{169}$$

rescribed flow rate: $Q = Byv$

The second used equation is:

$$Q = B\frac{c^2}{y}v \tag{170}$$

th prescribed level:

See 19.

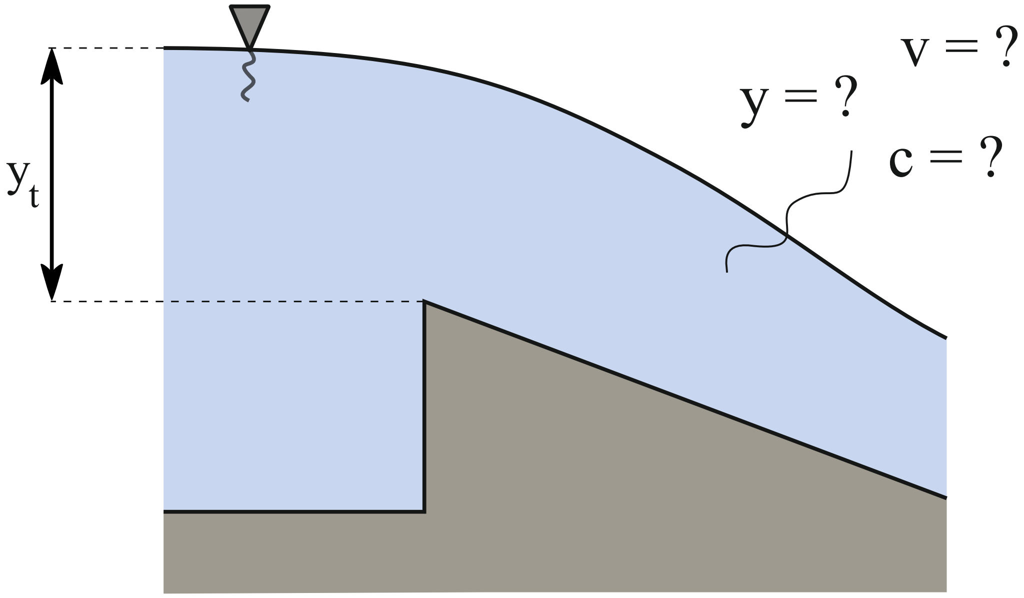


Figure 19: Tank with prescribed level.

Bernoulli's equation:

$$P_t = P + \frac{\rho}{2} v^2 \tag{171}$$

$$y_t \rho g = y \rho g + \frac{\rho}{2} v^2 \tag{172}$$

$$y_t = y + \frac{v^2}{2g} \tag{173}$$

Also we have to decide the flow direction.

$$y_t - \frac{c^2}{g} - \frac{v^2}{2g} = 0 \tag{174}$$

$$y_t - \frac{1}{g} (\frac{v - K_R}{2})^2 - \frac{v^2}{2g} = 0 \tag{175}$$

$$-\frac{3}{4}v^2 + \frac{K_R}{2}v + gy_t - \frac{K_R^2}{4} = 0 \tag{176}$$

Thus, we obtain v:

$$v_{1,2} = \frac{1}{2(-\frac{3}{4})}\left(-\frac{K_R}{2} \pm \sqrt{\frac{K_R^2}{4} - 4(-\frac{3}{4})(gy_t - \frac{K_R^2}{4})}\right) = \frac{2}{3}\left(\frac{K_R}{2} \pm \sqrt{\frac{K_R^2}{4} + 3(gy_t - \frac{K_R^2}{4})}\right) \tag{177}$$

$$v_{1,2} = \frac{K_r}{3} \pm \sqrt{3gy_t - \frac{1}{2}K_R^2}\frac{2}{3} \tag{178}$$

When $v > 0$ does occur:

$$\frac{2}{3}\sqrt{3gy_t - \frac{1}{2}K_R^2} = \frac{K_R}{3} \tag{179}$$

$$3gy_t - \frac{1}{2}K_R^2 = \frac{K_R^2}{4} \tag{180}$$

$$3gy_t = \frac{3}{4}K_R^2 \tag{181}$$

The second used equation is:

$$gy_t = \left(\frac{K_R}{2}\right)^2 \tag{182}$$

If $K_R > 2\sqrt{gy_t}$, we have inflow ($v > 0$). Note that Fr¿1 at the node is not physical(see 20).

If $v > c$ occurs (see 21), we have to override according to the followings: The first used equation instead of $v - 2c = K_R$ is $v = c$. The second used equation is $y_t = y + \frac{v^2}{2g}$, thus:

$$y_t = y + \frac{gy}{2g} = \frac{3}{2}y \rightarrow y = \frac{2}{3}y_t \tag{183}$$

Summary

Similar as for pressurized flow if $F_r < 1$, however, for $F_r > 1$ boundary condition is needed. For supercritical inflow both y and v or $c$and v can be prescribed.

Right boundary

The Right boundary condition can be determined similarly to the Left boundary condition, using α characteristic instead of β, thus the first equation is $v + 2c = \alpha_L + \Delta t g(S_0 - S_f) = K_L$.

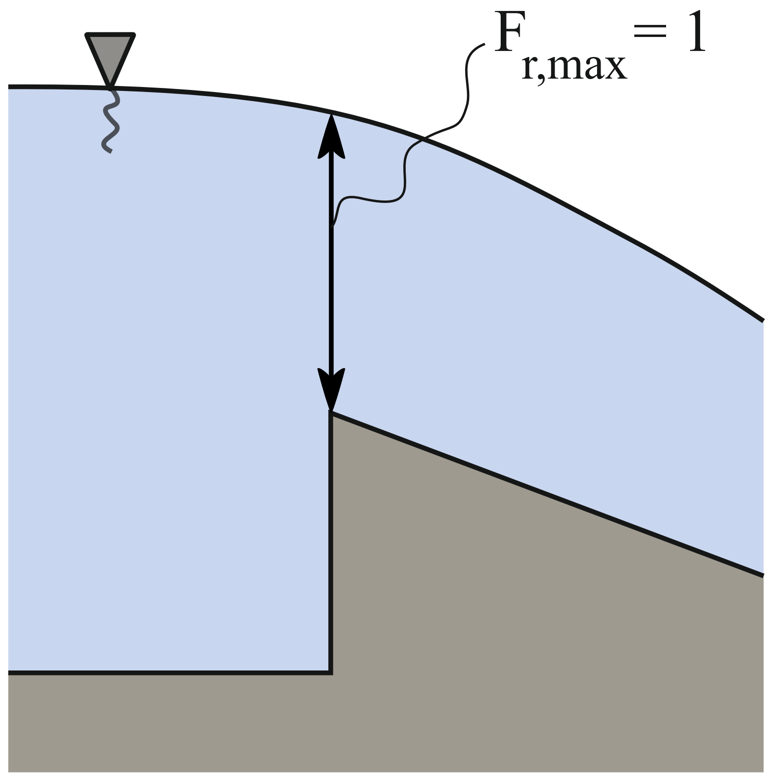


Figure 20: $Fr_max = 1$.

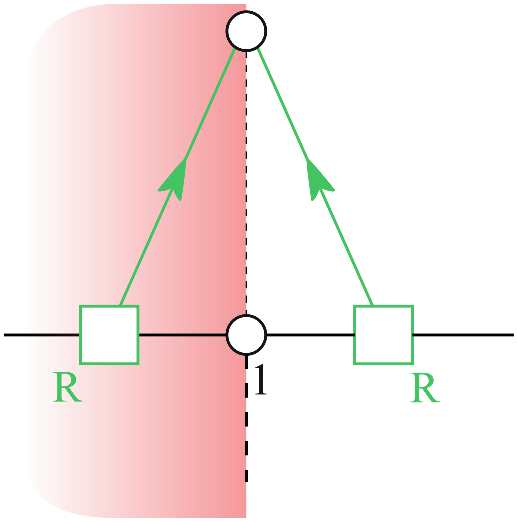


Figure 21: Domain in case of $v > c$.

6 Unsteady 1D compressible gas flow

6.1 Governing equations

Let us start off with the 1D integral form of the continuity equation

$$\frac{\partial}{\partial t}\int_V \rho dV + \oint_A \rho \underline{v}\, d\underline{A} = 0, \tag{184}$$

equation of motion

$$\frac{\partial}{\partial t}\int_V \rho \underline{v} dV + \oint_A \underline{v}\rho\underline{v}\, d\underline{A} = -\oint_A p\, d\underline{A} + \oint_A \underline{\underline{\tau}}\, d\underline{A}, \tag{185}$$

and energy equation

$$\frac{\partial}{\partial t}\int_V \rho e dV + \oint_A e\rho\underline{v}\, d\underline{A} = -\oint_A p\underline{v}\, d\underline{A} + \oint_A \underline{q}\, d\underline{A} \tag{186}$$

where $\underline{\underline{\tau}}$ is the stress tensor, $e = u + v^2/2$ (for an ideal gas, we have $du = c_V dT$) is the sum of internal energy and kinetic energy and $\underline{q}$ is the heat flux vector. We also need an equation of state of the form $p = f(\rho, u)$.

In what follows, we assume 1D flow, hence $\underline{v} = v(x,t)$.

Applying the divergence theorem on the continuity equation and expoiting that $V = A(x)x$, we have

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_V \frac{\partial}{\partial x}(\rho\underline{v})\, dV = \int \left(\frac{\partial\, \rho A}{\partial t} + \frac{\partial\, \rho v A}{\partial x}\right) dx = 0. \tag{187}$$

$$\frac{\partial\, \rho A}{\partial t} + \frac{\partial\, \rho v A}{\partial x} = 0. \tag{188}$$

The equation of motion takes the form

$$\frac{\partial}{\partial t}\int_V \rho v dV + \int_V \frac{\partial}{\partial x}\rho v^2\, dV = -\int \frac{\partial p}{\partial x} A dx + \oint_A \underline{\underline{\tau}}\, d\underline{A} \tag{189}$$

$$\int \left(\frac{\partial}{\partial t}\rho v A + \frac{\partial}{\partial x} A\rho v^2\right) dx = -\int \left(\frac{\partial (pA)}{\partial x} - p\frac{dA}{dx}\right) dx + \int F_s\, dx \tag{190}$$

$$\frac{\partial}{\partial t}\rho v A + \frac{\partial}{\partial x}\left(A\rho v^2 + pA\right) = p\frac{dA}{dx} + F_s = F_p + F_s. \tag{191}$$

The energy equation can be rewritten in a similar way. Finally, the system of equation to be solved are the continuity equation (192), equation of motion (193), energy equation (194) and some type of material law (for e.g. isentropic flow) (195)

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho v A) = 0 \tag{192}$$

$$\frac{\partial}{\partial t}(\rho v A) + \frac{\partial}{\partial x}\left(A\rho v^2 + pA\right) = F_p + F_s \tag{193}$$

$$\frac{\partial}{\partial t}(\rho e A) + \frac{\partial}{\partial x}(A\rho v e + p v A) = \dot{Q} \tag{194}$$

$$\frac{p}{\rho} = RT \quad \text{(or some other material equation)} \tag{195}$$

Some praticulat behaviour might be assumed, such that isentropic, isotherm, polytropic or else. The unknowns of the equation are $\rho(x,t), v(x,t), p(x,t), T(x,t)$.

Isentropic MOC	Lax-Wendroff scheme
adiabatic and reversible, thus no friction losses ⊖	any physical effect can be handled on the RHS ⊕
no heat exchange ⊖	BC hard handling with MOC ⊖
shock waves are naturally included ⊕	needs artificial damping around shock waves ⊖

Table 1: Caption!

6.2 Isentropic MOC (Method of Characteristics

Let us start off with some basic equations from thermodynamics. We assume an isentropic flows, thus there is no heat transfer and the $\dot{Q}$ is zero. The pipe is perfectly isolated and the processes are reversible so that means it is lossless, thus the frictional terms are zero. From the ideal gas law and the isentropic process, we have

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{1}{\kappa-1}} = \left(\frac{a}{a_0}\right)^{\frac{2}{\kappa-1}}, \tag{196}$$

$$\frac{\partial \rho}{\partial x} = \rho_0 \frac{2}{\kappa-1}\left(\frac{a}{a_0}\right)^{\frac{2}{\kappa-1}-1} \frac{\partial a}{\partial x}\frac{1}{a_0} = \rho_0 \frac{2}{\kappa-1}\frac{\rho}{\rho_0}\left(\frac{a}{a_0}\right)^{-1}\frac{\partial a}{\partial x}\frac{1}{a_0} = \frac{2}{\kappa-1}\frac{\rho}{a}\frac{\partial a}{\partial x}, \tag{197}$$

$$\frac{\partial \rho}{\partial t} = \frac{2}{\kappa-1}\frac{\rho}{a}\frac{\partial a}{\partial t} \quad \text{and} \tag{198}$$

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_{isentropic} \quad \rightarrow \quad \frac{\partial p}{\partial x} = a^2 \frac{\partial \rho}{\partial t}. \tag{199}$$

Now, the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = \frac{2}{\kappa-1}\frac{\rho}{a}\frac{\partial a}{\partial t} + \underbrace{v\frac{2}{\kappa-1}\frac{\rho}{a}\frac{\partial a}{\partial x}}_{v\frac{\partial \rho}{\partial x}} + \rho\frac{\partial v}{\partial x} \tag{200}$$

$$= \frac{2}{\kappa-1}\frac{\rho}{a}\left(\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x}\right) + \rho\frac{\partial v}{\partial x} = \frac{2}{\kappa-1}\frac{\rho}{a}\frac{\mathrm{d}a}{\mathrm{d}t} + \rho\frac{\partial v}{\partial x} = 0 \tag{201}$$

and the equation of motion becomes

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\mathrm{d}v}{\mathrm{d}t} + a\frac{2}{\kappa-1}\frac{\partial a}{\partial t} = 0 \tag{202}$$

Computing $\frac{a}{\rho}\frac{\kappa-1}{2}$ (201) $+\frac{\kappa-1}{2}$ (202) gives

$$0 = \frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\kappa-1}{2}a\frac{\partial v}{\partial x} + \frac{\kappa-1}{2}\frac{\mathrm{d}v}{\mathrm{d}t} + a\frac{\partial a}{\partial t} = \left(\frac{\mathrm{d}a}{\mathrm{d}t} + a\frac{\partial a}{\partial x}\right) + \frac{\kappa-1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + a\frac{\partial v}{\partial x}\right) \tag{203}$$

$$= \left(\frac{\partial a}{\partial t} + (a+v)\frac{\partial a}{\partial x}\right) + \frac{\kappa-1}{2}\left(\frac{\partial v}{\partial t} + (a+v)\frac{\partial v}{\partial x}\right) \tag{204}$$

$$:= \frac{\mathcal{D}^+ a}{\mathcal{D}^+ t} + \frac{\kappa-1}{2}\frac{\mathcal{D}^+ v}{\mathcal{D}^+ t} = \frac{\mathcal{D}^+}{\mathcal{D}^+ t}\left(a + \frac{\kappa-1}{2}v\right) := \frac{\mathcal{D}^+\alpha}{\mathcal{D}^+ t} \tag{205}$$

A similar computation with $\frac{a}{\rho}\frac{\kappa-1}{2}$ (201) $-\frac{\kappa-1}{2}$ (202) gives

$$0 = \frac{\mathcal{D}^-\beta}{\mathcal{D}^- t} \quad \text{with} \quad \frac{\mathcal{D}^-}{\mathcal{D}^- t} = \frac{\partial}{\partial t} + (v-a)\frac{\partial}{\partial x} \quad \text{and} \quad \beta = a - \frac{\kappa-1}{2}v \tag{206}$$

6.3 Boundary condition handling of the pipe model

6.3.1 General framework

The boundary conditions are implemented with the help of isentropic method of characteristics. Upon assuming adiabatic frictionless flow (i.e. neglecting heat flux through the pipe wall and any additional loss), system can be rewritten as

$$0 = \frac{\mathcal{D}^+}{\mathcal{D}^+ t}\left(a + \frac{\kappa-1}{2}v\right) := \frac{\mathcal{D}^+\alpha}{\mathcal{D}^+ t} \quad \text{along the curve} \qquad \frac{dx}{dt} = v + a \tag{207}$$

$$0 = \frac{\mathcal{D}^-}{\mathcal{D}^- t}\left(a - \frac{\kappa-1}{2}v\right) := \frac{\mathcal{D}^+\beta}{\mathcal{D}^+ t} \quad \text{along the curve} \qquad \frac{dx}{dt} = v - a \tag{208}$$

This means that along two special curves $dx/dt = v \pm a$ the PDE system can be decoupled to two uncoupled ordinary differential equations. The notation $\frac{\mathcal{D}^\pm \bullet}{\mathcal{D}^\pm t}$ denotes differentiation along these two directions with slopes $1/(v \pm a)$, respectively.

However, it is nontrivial to decide that for a given tank pressure p_t the flow is inwards/outwards and subsonic/supersonic.

6.3.2 Inflow

Consider first the case of inflow, that is, one can compute the location ξ_P, from which the $\mathcal{C}^-$ characteristics runs to $(\xi, \tau) = (0, \Delta t)$, as depicted in panel (a) of Figure 22. Let $\tilde{a}$ and $\tilde{v}$ denote the unknown values at $(\xi, \tau) = (0, \Delta t)$. We have

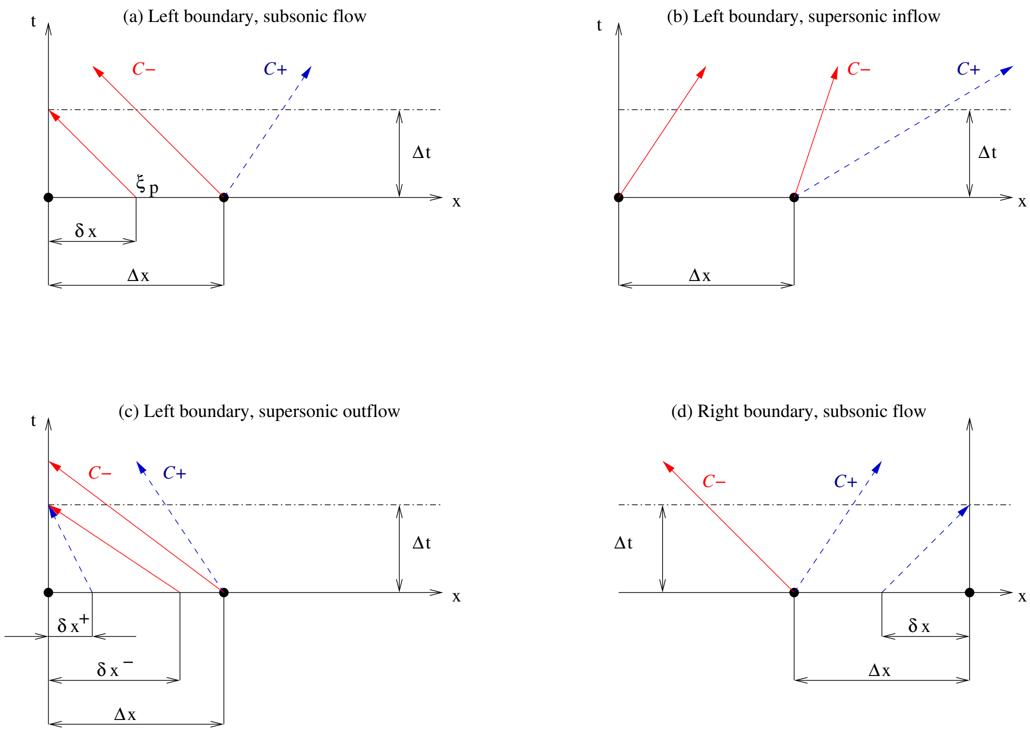


Figure 22: Boundary conditions for the isentropic MOC.

$$\tilde{a} - \frac{\kappa - 1}{2}\tilde{v} = \beta_P. \tag{209}$$

Assuming isentropic inflow from the tank to the pipe, we also have

$$c_p\tilde{T} + \frac{\tilde{v}^2}{2} = c_p T_t, \tag{210}$$

thus

$$\beta_P = \tilde{a} - \frac{\kappa - 1}{2}\tilde{v} = \sqrt{\kappa R\left(T_t - \frac{\tilde{v}^2}{2c_p}\right)} - \frac{\kappa - 1}{2}\tilde{v}, \tag{211}$$

which is a single equation for $\tilde{v}$. The solution is

$$\tilde{v} := \tilde{v}_{in} = \frac{2}{\kappa + 1}\left(-\beta_P + \sqrt{a_t^2\frac{\kappa + 1}{\kappa - 1} - \frac{2}{\kappa - 1}\beta_P^2}\right) \tag{212}$$

and it is straightforward to check that $\tilde{v}_{in} > 0$ iff $\beta_P < a_t$.

The next step is to find the critical β_P value at which the flow becomes sonic, i.e. $\tilde{v}_{in} = \tilde{a}$. It turns out that there are *two* solutions:

$$\beta_{P,s,1} = \pm a_t \sqrt{\frac{\kappa+1}{2}} \quad \text{and} \quad \beta_{P,s,2} = \pm a_t \frac{3-\kappa}{\sqrt{2(\kappa+1)}}, \tag{213}$$

out of which only one corresponds to sonic inflow, i.e. $\beta_{P,s} = a_t \frac{3-\kappa}{\sqrt{2(\kappa+1)}}$ and we have

$$\tilde{v}_{in,max} = \sqrt{a_t}\sqrt{\frac{2}{1+\kappa}}. \tag{214}$$

This is a special case of panel (b) in Figure 22, when the $\mathcal{C}^-$ characteristic line at $x = 0$ becomes vertical. Note that in our case (tank-to-pipe flow), the inflow cannot be supersonic as a convergent-divergent channel (i.e. a Laval nozzle) would be needed to reach this state.

6.3.3 Outflow

In this case, the characteristic equation is still valid, yet the outflow is not isentropic as the kinetic energy of the gas $\tilde{v}^2/2$ is lost. Instead, we can prescribe tank pressure at $\xi = 0$, unless the flow is subsonic - panel (a) in Figure 22. We have

$$\tilde{a} - \frac{\kappa-1}{2}\tilde{v} = \beta_P \quad \text{and} \quad \tilde{p} = p_t. \tag{215}$$

Moreover, we use isentropic relationships from point ξ_P to $\xi = 0$. The velocity is

$$v_{out} = \frac{2}{\kappa-1}\left(-\beta_P + \sqrt{\left(\frac{p_t}{p_P}\right)^{\frac{\kappa-1}{\kappa}}} a_P\right) = \frac{2}{\kappa-1}\left(-\beta_P + a_t\right) < 0 \tag{216}$$

Sonic outflow is reached if $\beta_P = \frac{\kappa+1}{2}a_t$, which is

$$a_t = a_P \left(\frac{p_t}{p_P}\right)^{\frac{\kappa-1}{2\kappa}} \tag{217}$$

and hence $v_{out,max} = -a_t$. Again, supersonic outflow - panel (c) in Figure 22 - cannot be reached in this simple configuration.

Finally, we have the following cases.

$$v_{in} = \begin{cases} \sqrt{a_t}\sqrt{\frac{2}{\kappa+1}} & \text{if} \quad \beta_P/a_t < \frac{\kappa+1}{2} \\ \frac{2}{\kappa+1}\left(-\beta_P + \sqrt{a_t^2 \frac{\kappa+1}{\kappa-1} - \frac{2}{\kappa-1}\beta_P^2}\right) & \text{if} \quad \frac{\kappa+1}{2} < \beta_P/a_t < 1 \\ \frac{2}{\kappa+1}\left(-\beta_P + a_t\right) < 0 & \text{if} \quad 1 < \beta_P/a_t < \frac{3-\kappa}{\sqrt{2(2+\kappa)}} \\ -a_t & \text{if} \quad \frac{3-\kappa}{\sqrt{2(2+\kappa)}} < \beta_P/a_t \end{cases}$$

For the sake of completeness, we also depicted the subsonic case at the valve-end of the pipe in panel (d) of Figure 22.

6.4 Finite Differencies for hyperbolic conservation laws

A scalar hyperbolic conservation law is:

$$\frac{\partial u}{\partial t}+\frac{\partial f\left(u\right)}{\partial x} \tag{218}$$

where u is the conserved quantity and f is the flux. Now, let compute the rate of u in the internal $a\leq x\leq b$:

$$\frac{\partial}{\partial t}\int_{a}^{b}u\left(x,t\right)dx=\int_{a}^{b}\frac{\partial u}{\partial t}dx=\dots \tag{219}$$

We see that the rate of the change equals to the fluxes at boundaries. In other words, if $f\left(a\text{or}b,t\right)\equiv0$ then the $\int_{a}^{b}udx$ remains *constant* = *conserved*. By applying the chain rule, we obtain:

$$\frac{\partial u}{\partial t}+\underbrace{\frac{\partial f}{\partial u}}_{Jacobian}\frac{\partial u}{\partial c}=0 \tag{220}$$

In the special case of $\frac{df}{du}=\lambda\quad(constant)$, then we have the following wave equation:

$$\frac{\partial u}{\partial t}+\lambda\frac{\partial u}{\partial x}=0 \tag{221}$$

6.4.1 Eplicit in time, centered in space

IDE KEPET KELL BESZURNI

$$\frac{u_{i}^{j+1}-u_{i}^{j}}{\Delta t}=-\frac{f_{i+1}^{j}-f_{i-1}^{j}}{2\Delta x} \tag{222}$$

The error is proportional with Δt and Δx^{2} and overall the error will be first order.

6.4.2 Upwinding

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$$\frac{u_{i}^{j+1}-u_{i}^{j}}{\Delta t}=\begin{cases}-\frac{f_{i}^{j}-f_{i-1}^{j}}{\Delta x}\\ -\frac{f_{i+1}^{j}-f_{i}^{j}}{\Delta x}\end{cases}\quad\text{error proportional with}\Delta t,\Delta x \tag{223}$$

6.4.3 Implicit in time, centered in space

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$$\frac{u_{i}^{j+1}-u_{i}^{j}}{\Delta t}=-\frac{f_{i+1}^{j+1}-f_{i-1}^{j+1}}{2\Delta x} \tag{224}$$

We end up with a system of coupled nonlinear algebric equations:

$$u_{i}^{j+1}=u_{i}^{j}-\frac{\Delta t}{2\Delta x}\left(f\left(u_{i+1}^{j+1}\right)-f\left(u_{i-1}^{j+1}\right)\right)\text{for}\quad i=2\dots N-1+\text{BC's for}\quad i=1,i=N! \tag{225}$$

- Δt and Δx are *not coupled* (as in MOC). Still the CFL $\Delta t < \min\left(\frac{\Delta x}{a+|v|}\right)$ criteria is useful

- implicit method is stable, but does not imply accuracy and vica versa

- have to solve the equation: $\underline{G}\left(u^{j+1}\right) = \underline{0} \rightarrow$ need for $\underline{u}_0^{j+1}$ initial point

  1. use the old value: $\underline{u}^j$
  2. another trick: use the result of an explicit step such as: $u_{0,i}^{j+1} = u_i^j - \frac{\Delta t}{2\Delta x}\left(f\left(u_{j+1}^j\right) - f\left(u_{i-1}^j\right)\right)$ Predictor step

6.4.4 The Lax-Wendroff scheme

The governing equations can be written in a compact form

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = \mathcal{Q}, \tag{226}$$

with

$$\mathcal{U} = \begin{pmatrix} \rho A \\ \rho v A \\ \rho e A \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} \rho v A \\ \left(\rho v^2 + p\right) A \\ \left(\rho e v + p v\right) A \end{pmatrix}, \quad \text{and} \quad \mathcal{U} = \begin{pmatrix} 0 \\ F_p + F_s \\ \dot{Q} \end{pmatrix}. \tag{227}$$

Here the internal energy $e = c_V T$, $F_p = p\frac{\mathrm{d}A(x)}{\mathrm{d}x}$ and $F_s = A\frac{\rho}{2}\frac{\lambda}{D}v|v|$.

Note that if $\mathcal{U}$ is known, the primitive valiables can also be computed: $\rho = \mathcal{U}_1/A$, $v = \mathcal{U}_2/\mathcal{U}_1$ and $e = \mathcal{U}_3/\mathcal{U}_1$.

The LaxWendroff method, named after Peter Lax and Burton Wendroff, is a numerical method for the solution of hyperbolic partial differential equations, based on finite differences. It is second-order accurate in both space and time. The sceme is depicted in Figure 23 and consists of the following steps.

1. Update $\mathcal{U}_{j+\frac{1}{2}}^{i+\frac{1}{2}}$ at the half time level and middle grid points from

$$\frac{\mathcal{U}_{j+\frac{1}{2}}^{i+\frac{1}{2}} - \mathcal{U}_{j+\frac{1}{2}}^{i}}{\Delta t/2} + \frac{\mathcal{F}_{j+1}^{i} - \mathcal{F}_{j}^{i}}{\Delta x} = \frac{\mathcal{Q}_{j+1}^{i} + \mathcal{Q}_{j}^{i}}{2}, \quad \text{where} \quad \mathcal{U}_{j+\frac{1}{2}}^{i} = \frac{\mathcal{U}_{j+1}^{i} + \mathcal{U}_{j}^{i}}{2}. \tag{228}$$

2. 'Unpack' the primitive valiables and compute $\mathcal{F}_{j+\frac{1}{2}}^{i+\frac{1}{2}}$.

3. Take a full time step to compute $\mathcal{U}_j^{i+1}$ with the help of $\mathcal{F}_{j+\frac{1}{2}}^{i+\frac{1}{2}}$:

$$\frac{\mathcal{U}_j^{i+1} - \mathcal{U}_j^{i}}{\Delta t} + \frac{\mathcal{F}_{j+\frac{1}{2}}^{i+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}}^{i+\frac{1}{2}}}{\Delta x} = \frac{\mathcal{Q}_{j+\frac{1}{2}}^{i+\frac{1}{2}} + \mathcal{Q}_{j-\frac{1}{2}}^{i+\frac{1}{2}}}{2}. \tag{229}$$

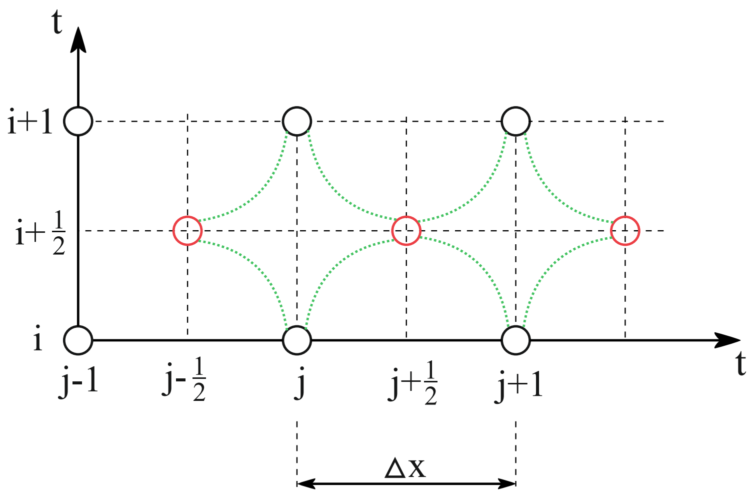

Figure 23: The Lax-Wendroff scheme.

The time step Δt cannot be chosen arbitrarily, the timestep should be small enough to ensure that the scheme does not 'steps over' a cell with information propagation velocity $a + |v|$:

$$\Delta t_j < C \frac{\Delta x}{a_j + |v_j|}, \quad \Delta t = \min \Delta t_j, \tag{230}$$

where $C < 1$ is the 'safety' factor and $a_j = \sqrt{\kappa R T_j}$.

7 Calculation of the wave speed under different circumstances

7.1 Wave, phase velocity, group velocity

An H_0 amplitude, moving wave described as,

$$H(x,t) = H_0 cos(kx - \omega t), \tag{231}$$

Here k is the wavenumber, ω is the angular frequency. These could described by the λ wavelength and the ν frequency: $k = \frac{2\pi}{\lambda}$; $\omega = 2\pi\nu$. Twice derivation of the equation (231) shows that $H(x,t)$ satisfy the $\frac{\partial^2 H}{\partial t^2} - \left(\frac{\omega}{k}\right)^2 \frac{\partial^2 H}{\partial x^2} = 0$ wave equation. The $\frac{\omega}{k} = a_p$ quotient called *phase velocity* and the $H(x,t)$ wave moving with this speed. Obviously $a_p = \nu\lambda$ conditions are met.

In case if ω angular frequency is not constant but also the k wavenumber is the function of $\omega(k)$, then the emerging as more elementary wave superposition of wave packet rate is differ from some of the component of the phase velocity.

The velocity of the wave packet called as group velocity and marked with a_g.

$$a_g = \frac{d\omega(k)}{dk} \tag{232}$$

Actually, it is two slightly perturbated wave sum,

$$cos((k - \Delta k)x - (\omega - \Delta\omega)t) + cos((k + \Delta k)x - (\omega + \Delta\omega)t) = 2cos(\Delta kx - \Delta\omega t)\, cos(kx - \omega t) \tag{233}$$

which amplitude is not yet constant, but also is a harmonic function. The phase velocity of amplitude is $\frac{\Delta\omega}{\Delta k}$, and the limit value of it is $\frac{d\omega}{dk}$ called group velocity.

7.2 Wave velocity of fluid flowing through pipe

1D equation of motion of slightly compressible fluid in varying cross section pipe

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho A v)}{\partial x} = \frac{\partial(\rho A)}{\partial t} + v\frac{\partial(\rho A)}{\partial x} + \rho A\frac{\partial v}{\partial x} = 0 \tag{234}$$

In the 2 first term the derivative is convertible if the multiplication $(\rho \cdot A)$ only depends on the pressure (status indicator's function)

$$\frac{d(\rho A)}{dp}\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \rho A\frac{\partial v}{\partial x} = 0 \tag{235}$$

$$\frac{d(\rho A)}{dp} = A\frac{d\rho}{dp} + \rho\frac{dA}{dp} = A\frac{\rho}{E_f} + \rho\frac{dA}{dp} \tag{236}$$

Write the change of the cross section caused by pressure changing in circular cross section, thin walled pipe! Initially the well known furnace formula and the Hook's law is

$$D \cdot dp = 2\delta \cdot \sigma d\sigma = \frac{dK}{K}E_{pipe} \tag{237}$$

Here δ is the thickness of the pipe, D is the inner diameter, σ is the wall tension, K is the perimeter of the pipe, E_{pipe} is the Young modulus of the pipe wall.

The cross section of the pipe is

$$A = \frac{D^2\pi}{4} = \frac{D^2\pi^2}{4\pi} = \frac{K^2}{4\pi}. \tag{238}$$

Derived from the above shown equation

$$\frac{dA}{dK} = \frac{2K}{4\pi} = \frac{2D\pi}{4\pi} \cdot \frac{D}{D} = \frac{2A}{K} \tag{239}$$

and after a bit equalization we get the following

$$\frac{dK}{K} = \frac{dA}{2A} \tag{240}$$

Now substitute the derived equation into the (237) follows to

$$d\sigma = \frac{dA}{A}\frac{E_{pipe}}{2} \tag{241}$$

thus

$$D \cdot dp = \delta \cdot 2d\sigma = \sigma \cdot \frac{dA}{A}E_{pipe} \tag{242}$$

and

$$\frac{dA}{dp} = A\frac{D}{\delta \cdot E_{pipe}} \tag{243}$$

Finally the equation of the continuity is derived

$$\left(A\frac{\rho}{E_f} + \rho A\frac{D}{\delta E_{pipe}}\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \rho A\frac{\partial v}{\partial x} = 0 \tag{244}$$

Simplifying by the A cross section value and introducing the E_r reduced Young modulus will produce the followings

$$\rho\left(\frac{1}{E_f} + \frac{D}{\delta E_{pipe}}\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \rho\frac{\partial v}{\partial x} = \frac{\rho}{E_r}\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \rho\frac{\partial v}{\partial x} = 0 \tag{245}$$

The wave speed in the simplest case, *in circular thin walled pipes* is

$$a = \sqrt{\frac{E_r}{\rho}}, \quad \frac{1}{E_r} = \frac{1}{E_{fluid}} + \frac{D}{\delta E_{pipe}} \tag{246}$$

if the fluid streaming in the pipe is slightly compressible. Here ρ is the fluid density, D and δ are the inner pipe diameter and pipe wall thickness, E_{fluid} is the bulk modulus of the fluid, E_{pipe} is the elasticity modulus of the pipe wall material.

Compared to this basic situation the different *longitudinal supports or fixing* of the pipe will cause changes. The Poisson-number μ is the ratio of the longitudinal and circumferential stresses, its value for thin pipe wall is 0.5. The influence of the support can be considered by a factor n.

$$a = \sqrt{\frac{E_{reduced}}{\rho}}, \quad \frac{1}{E_{reduced}} = \frac{1}{E_{fluid}} + n\frac{D}{\delta E_{pipe}}. \tag{247}$$

For thin walled pipes $n = n_{thin}$ where

- axial support of the pipe at both ends: $n_{thin} = 1 - \mu/2$,
- longitudinal support throughout the pipe length: $n_{thin} = 1 - \mu^2$,
- without support, naturally: $n_{thin} = 1$.

If compared to the diameter the wall is thick the value of n will be calculated from the above value of n_{thin}:

$$n = \frac{2\delta}{D}(1+\mu) + \frac{D}{D+\delta}n_{thin}. \tag{248}$$

For gas flow

$$a = \sqrt{\frac{dp}{d\rho}}, \tag{249}$$

where assuming ideal gas $dp/d\rho = \kappa RT$, thus $a = \sqrt{\kappa RT} = \sqrt{\kappa p/\rho}$, which is the well-known isothermal speed of sound. As $E/\rho = a^2 = dp/d\rho = \kappa p/\rho$, obviously for gases

$$E_{gas} = \kappa p. \tag{250}$$

For open channel flow instead of $dp/d\rho$, the actual coefficient in the continuity equation is $dy/dA = 1/B$. Multiplying the equation

$$B\left(\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x}\right) + A\frac{\partial v}{\partial x} = 0 \tag{251}$$

with ρ/A and introducing the gravitational acceleration g:

$$\frac{B}{Ag}\left(\frac{\partial \rho g y}{\partial t} + v\frac{\partial \rho g y}{\partial x}\right) + \rho\frac{\partial v}{\partial x} = 0. \tag{252}$$

This equation has the same dimension as the earlier continuity equations from which $B/(Ag) = 1/a^2$, thus the free surface wave speed is

$$a = \sqrt{\frac{Ag}{B}}. \tag{253}$$

In case if the pipe is completely full in the whole cross section the equation (253) gives infinite wave velocity because the cross section is finite and B goes to zero. The usual solution such a case that along the upper constituent of the pipe a narrow and parallel types wall was placed which width is B_S which is just as large as that the fulled pipe cross section equation's wave velocity will given.

7.3 Wave velocity in case of gas flow

$$a = \sqrt{\frac{dp}{d\rho}} \tag{254}$$

where we assume ideal gas properties $\frac{dp}{d\rho} = \kappa RT$, thus

$$a = \sqrt{\kappa RT} = \sqrt{\kappa \frac{p}{\rho}} \tag{255}$$

which actually is the well known isentropic moving velocity of sound wave. And

$$\frac{E}{\rho} = a^2 = \frac{dp}{d\rho} = \kappa \frac{p}{\rho} \tag{256}$$

thus in case of gase

$$E_{gas} = \kappa p \tag{257}$$

7.4 Wave and group velocity in elastic quadrangular cross sectional channel

In air conditioning the ventilation channels often have rectangular cross section bended from thin metal plates. These, 1-2 m long channel segments are fixed to each other. Experiments with standing waves have proved that in this case the wave speed a strongly depends on the angular frequency ω of the pressure wave. This leads to its dispersion. The $a(\omega)$ function has the form:

$$a(\omega) = \frac{c}{\sqrt{1 + \rho c^2 f(\omega)}}, \tag{258}$$

where c is the isentropic sound velocity in free air, the function $f(\omega)$ depends on the geometry and material properties of the channel. L is the dimension of the channel side, δ is the thickness of channel wall:

$$f(\omega) = \frac{2L^3}{E_{wall}\delta^3\Omega^5}\left(\frac{2}{\cot\Omega + \coth\Omega} - \Omega\right), \quad \text{and} \quad \Omega = \sqrt[4]{\frac{3\rho L^4\omega^2}{4E_{wall}\delta^2}}. \tag{259}$$

The wave speed in liquids strongly depends on *free gas content*. The wave speed in gaseous fluids can drop up to a few $10m/s$-s although the wave speed in pure normal air is $340m/s$. The mixture characterized by void fraction $\alpha = V_g/V$ is composed of free gas of density ρ_g and fluid of density ρ_f (naturally $1 - \alpha = V_f/V$). The mass of the mixture is $\rho_g V_g + \rho_f V_f = \rho V$. Dividing this by the total volume V of the mixture and introducing the above abbreviation for the void fraction α the mean density is: $\rho = \alpha\rho_g + (1 - \alpha)\rho_f$. Now the wave speed must be calculated from the mean density and reduced elasticity modulus. The elasticity will also depend on the compressibility of the gas. By the definition of the elasticity modulus (Hooks law) $dV_g = -V_g/E_g dp$, and $dV_f = -V_f/E_f dp$. The total change of volume V is

$$dV = dV_g + dV_f = -\left(\frac{\alpha V}{E_g} + \frac{(1-\alpha)V}{E_f}\right) dp = -\left(\frac{\alpha}{E_g} + \frac{(1-\alpha)}{E_f}\right) V dp \tag{260}$$

and thus the reduced elasticity modulus E_e is

$$E_e = -V\frac{dp}{dV} = \frac{1}{\left(\frac{\alpha}{E_g} + \frac{(1-\alpha)}{E_f}\right)}. \tag{261}$$

Finally the square of the wave speed (considering that the extension of the pipe wall compared to the compressibility of gas content is negligible):

$$a^2 = \frac{E_e}{\rho} = \frac{1}{\left(\frac{\alpha}{E_g} + \frac{(1-\alpha)}{E_f}\right)(\alpha\rho_g + (1-\alpha)\rho_f)} = \frac{1}{\left(\frac{\alpha}{\kappa p} + \frac{(1-\alpha)}{E_f}\right)(\alpha\rho_g + (1-\alpha)\rho_f)}, \tag{262}$$

as $E_g = \kappa RT\rho = \kappa p$ (see above). This rather complicated formula can be simplified realizing that in the denominator in the first bracket the first, in the second bracket the second term is dominant

$$a = \sqrt{\frac{E_e}{\rho}} \approx \sqrt{\frac{\kappa p}{\alpha(1-\alpha)\rho_f}}. \tag{263}$$

Differentiating Eq. (263) with respect to α gives the minimum of the wave speed at $\alpha = 0.5$.

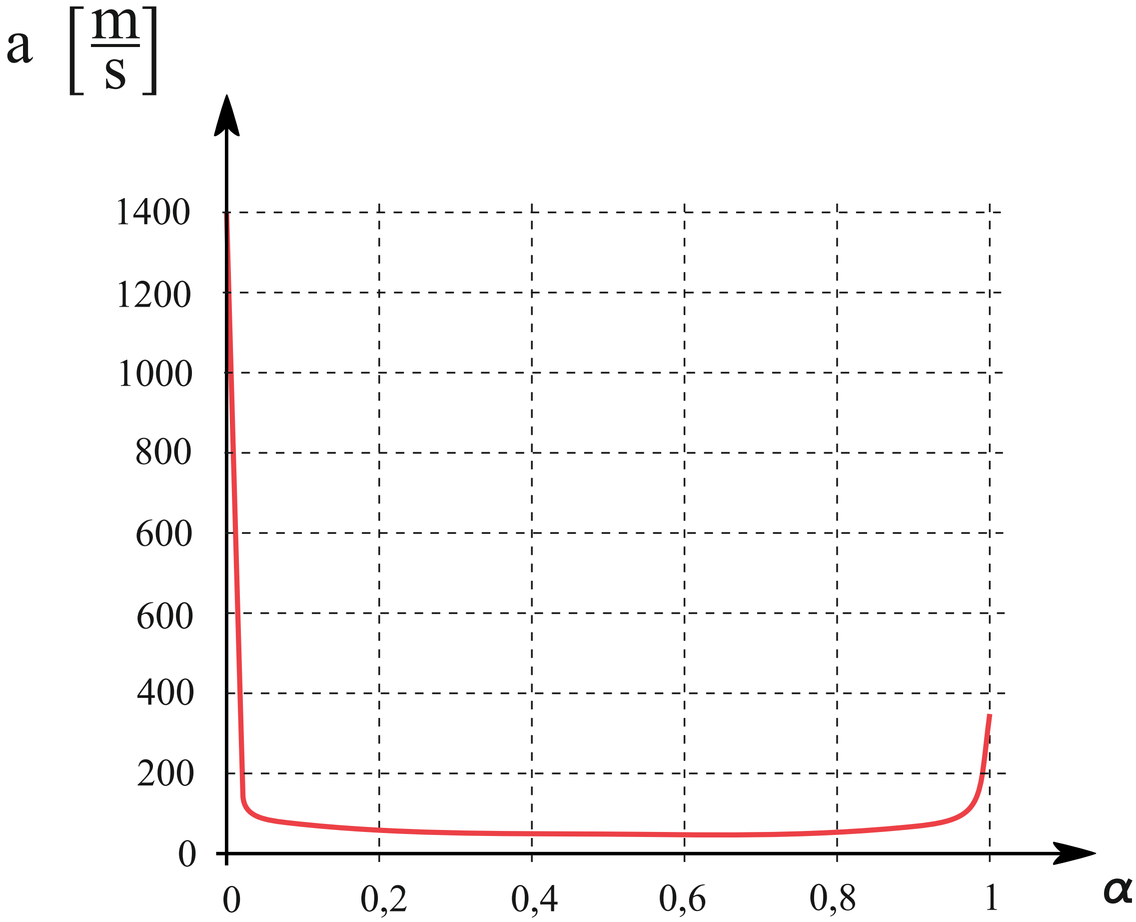

Figure 24: Wave speed in water containing air at 3 bar pressure

8 Model of a ventilation channel with rectangular cross-section

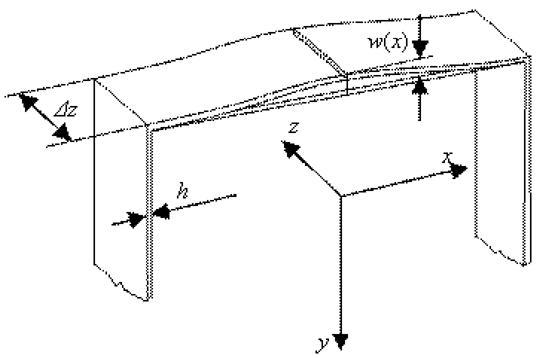

Figure 25: Wave speed in water containing air at 3 bar pressure

Euler-Bernoulli equation of the deflection of a beam (width: Δz, thickness: h) under uniform load in steady state:

$$EI_z \frac{d^4 w}{dx^4} = q \tag{264}$$

- E: elasticity modulus, $E = 2 \cdot 10^{11} Pa$ for steel
- I_z: area moment of inertia, $I_z = \frac{\Delta h^3}{12}$
- w: deflection of the longer channel side (u: deflection of the shorter channel side)
- x: longitudinal coordinate of the longer channel side (y: coordinate of the shorter channel side)

The above differential equation for *constant load* q (force per unit length, $q = dp\Delta z$ where dp is the change of overpressure in the channel) is:

$$EI_z \frac{d^4 w}{dx^4} = dp\Delta z \quad \text{or} \quad \frac{d^4 w}{dx^4} = \frac{dp\Delta z}{EI_z} \overset{\text{denoted}}{=} P. \tag{265}$$

The *general solution* of this differential equation is

$$w(x) = P\frac{x^4}{24} + C_3\frac{x^3}{6} + C_2\frac{x^2}{2} + C_1 x + C_0 \tag{266}$$

As we suppose *symmetry with respect to the centreline* of the beam (at $x = 0$) the terms of uneven order will drop:

$$w(x) = P\frac{x^4}{24} + C_2\frac{x^2}{2} + C_0 \tag{267}$$

and similarly

$$u(x) = P\frac{y^4}{24} + D_2\frac{y^2}{2} + D_0 \tag{268}$$

for the shorter side of the channel.

8.1 Boundary conditions

The corners are fixed: $w(L/2) = 0$ and $u(-B/2) = 0$. The tangents of the channel sides at the corner are perpendicular as the angle of the corner keeps its value 90°: $w''(L/2) = u''(-B/2)$. Denoting the *side ratio* of the rectangular cross section by $s = B/L$ the solutions satisfying the boundary conditions are

$$w(x) = P\frac{x^4}{24} + \frac{PL^2}{24}\left(2s^2 - 2s - 1\right)\frac{x^2}{2} + \frac{PL^4}{8\cdot 24}\left(\frac{1}{2} + 2s - 2s^2\right), \tag{269}$$

$$u(y) = P\frac{y^4}{24} + \frac{PL^2}{24}\left(-s^2 - 2s + 2\right)\frac{y^2}{2} + \frac{PL^4}{8\cdot 24}s^2\left(-2 + 2s + \frac{s^2}{2}\right). \tag{270}$$

The area change dA of the cross section $A = BH$ is the sum of the integrals of the deflections of the for side walls:

$$dA = 2\int_{x=-\frac{L}{2}}^{x=\frac{L}{2}} w(x)dx + 2\int_{y=-\frac{yL}{2}}^{y=\frac{sL}{2}} w(y)dy = \frac{PL^5}{24}\frac{s^5 + 5\left(s^4 - s^3 - s^2 + s\right) + 1}{15}. \tag{271}$$

If both the gas is compressible and the channel area is changing under the pressure change the equation of continuity is

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho A v)}{\partial x} = \frac{\partial(\rho A)}{\partial t} + v\frac{\partial(\rho A)}{\partial x} + \rho A\frac{\partial v}{\partial x} = \frac{d(\rho A)}{dp}\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \rho A\frac{\partial v}{\partial x} = 0. \tag{272}$$

After some calculations and introducing the isentropic wave velocity c in the gas we have

$$\frac{1}{\rho A}\left(A\frac{d\rho}{dp} + \rho\frac{dA}{dp}\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial v}{\partial x} = \left(\frac{1}{\rho c^2} + \frac{1}{A}\frac{dA}{dp}\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial v}{\partial x}$$

$$\overset{\text{denoted}}{=} \left(\frac{1}{\rho c^2} + \Phi\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial v}{\partial x} = \left(\frac{1}{\rho a^2}\right)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x}\right) + \frac{\partial v}{\partial x} = 0. \tag{273}$$

The wave velocity in the channel denoted above by a is

$$a = \frac{c}{\sqrt{1 + \rho c^2\Phi}} \quad \text{with} \quad \Phi = \frac{1}{A}\frac{dA}{dp} = \frac{L^3}{15Eh^3}\frac{s^5 + 5\left(s^4 - s^3 - s^2 + s\right) + 1}{2s}, \tag{274}$$

where ρ is the density of air at the given air temperature. Here $a = 173,4m/s$ if the air density is $\rho = 1,2kg/m^3$ and $c = 340m/s$ with other parameters given at the end of this section.

Experiments have not proved these formulae. Why? Because the channel side walls have a mass and one has to consider the inertia of this mass. The equation of *dynamic bending* of a slender isotropic homogeneous beam of constant cross section under a constant transverse uniform load is

$$EI_z\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = q(t) \overset{!}{=} \Delta z\cdot\hat{p}\cdot e^{\imath\omega t}. \tag{275}$$

Here m is the mass per unit length, $m = \rho_c h \Delta z$, ρ_c is the density of the channel wall and $d\hat{p} \cdot e^{\imath\omega t}$ is a harmonic excitation in the form of a complex function. We look for the general solution of this partial differential equation, PDE – fourth order in space, second order in time – as the product of a function depending only on space and another function depending only on time (this method is called Fourier decomposition):

$$w = \hat{w}(x) e^{\imath\omega t}. \tag{276}$$

After differentiation with respect to x and t, respectively and substituting into the PDE

$$E\frac{\Delta z h^3}{12}\frac{d^4\hat{w}}{dx^4} e^{\imath\omega t} + \rho_c h \Delta z (\imath\omega)^2 e^{\imath\omega t} = \Delta z d\hat{p} e^{\imath\omega t} \tag{277}$$

by dropping the exponential function and Δz both occurring in all terms and noticing that the square of the imaginary unit $\imath$ is $\imath^2 = -1$ and finally multiplying by $12/(Eh^3)$ one gets:

$$\frac{d^4\hat{w}}{dx^4} - \frac{12\rho_c}{Eh^2}\hat{w}\omega^2 = \frac{12 d\hat{p}}{Eh^3}. \tag{278}$$

The differential equation of the shorter side wall is similar:

$$\frac{d^4\hat{u}}{dy^4} - \frac{12\rho_c}{Eh^2}\hat{u}\omega^2 = \frac{12 d\hat{p}}{Eh^3}. \tag{279}$$

With the notation $K = \sqrt[4]{12\rho_c/(Eh^3)}$ the general solutions of these ODE-s are

$$\hat{w}(x) = Q\cosh\left(K\sqrt{\omega}x\right) + S\cos\left(K\sqrt{\omega}x\right) - \frac{12 d\hat{p}}{Eh^3K^4\omega^2} \tag{280}$$

and

$$\hat{u}(y) = R\cosh\left(K\sqrt{\omega}y\right) + T\cos\left(K\sqrt{\omega}y\right) - \frac{12 d\hat{p}}{Eh^3K^4\omega^2}. \tag{281}$$

The boundary conditions are identical with the previous ones. The solutions satisfying the boundary conditions using the side length ratio s again and denoting the constant term in the above differential equations by C result for the coefficients Q, S, R, T:

$$Q = -\frac{C}{\cosh\Omega}\frac{\tan\Omega + \tan(s\Omega)}{\tanh\Omega + \tan\Omega + \tanh(s\Omega) + \tan(s\Omega)} = -\frac{C}{\cosh\Omega}\mu, \tag{282}$$

$$R = -\frac{C}{\cosh(s\Omega)}\mu, \tag{283}$$

$$S = -\frac{C}{\cos\Omega}(1-\mu), \tag{284}$$

$$T = -\frac{C}{\cosh(s\Omega)}(1-\mu), \tag{285}$$

$$\tag{286}$$

with the notation $\Omega = K\sqrt{\omega}L/2$. Again the deflections of the four side walls can be integrated giving the change of area of channel cross section now depending on the excitation frequency ω. The final result for the function Φ is now

$$\Phi(\omega) = \frac{L^3}{15Eh^3s}\frac{45}{\Omega^5}\left(\frac{1}{\frac{1}{\tan\Omega+\tan(s\Omega)}+\frac{1}{\tanh\Omega+\tanh(s\Omega)}} - \Omega\frac{1+s}{2}\right) \tag{287}$$

and the wave velocity is

$$a = \frac{c}{\sqrt{1+\rho c^2\Phi(\omega)}}. \tag{288}$$

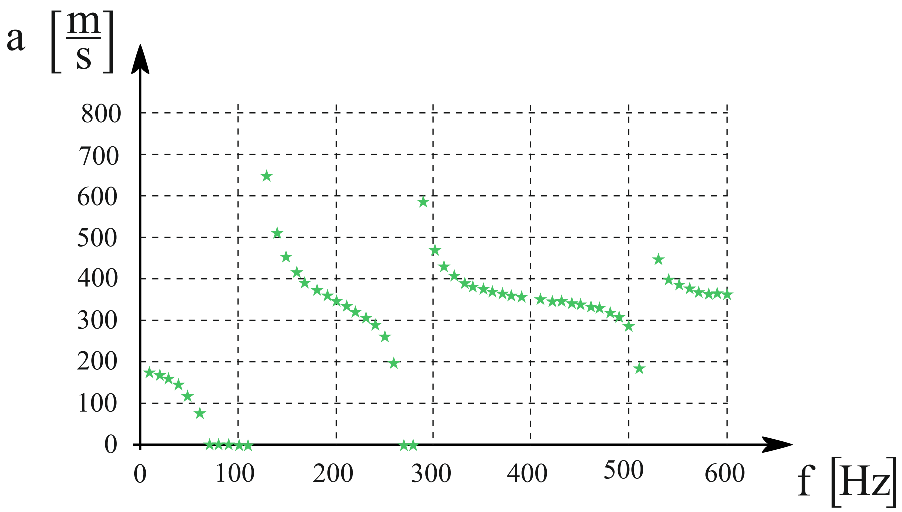

Figure 26: Wave velocity in a rectangular channel

9 Diffusion-convection-problem

9.1 Introduction

With notations C g/m^3 for the concentration of some solved material, w for the velocity vector of the fluid, t for time, ? for the diffusion coefficient:
convective flux (material-flow density):

$$q_c = wC \quad g/(m^2 s) \tag{289}$$

diffusive flux:

$$q_d = -\alpha gradC \quad g/(m^2 s) \tag{290}$$

The unsteady transport equation without sources is

$$\frac{\partial C}{\partial t} + divq_c = -divq_d = div(\alpha gradC) \tag{291}$$

If the diffusion coefficient doesnt depend on space then the RHS can be rewritten:

$$\frac{\partial C}{\partial t} + divq_c = -divq_d = \alpha div(gradC) = \alpha \triangle C \tag{292}$$

This last equation is in 1D flow in a pipe of constant cross section ($u = const$) is

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \alpha \frac{\partial^2 C}{\partial x^2} \tag{293}$$

x is the single space coordinate, u is the liquid velocity in x-direction.
In a $2D$ flow of an incompressible fluid ($divw = 0$) with the other coordinate y and liquid velocity component v:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = \alpha\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) \tag{294}$$

If solid particles are transported in settling basins with settling velocity vs a new source term appears on the RHS:

$$\frac{\partial C}{\partial t} + \frac{\partial (uC)}{\partial x} + \frac{\partial (vC)}{\partial y} = \alpha\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) + v_s\frac{\partial C}{\partial y} \tag{295}$$

Diffusion coefficients and solvability of some typical pollutants are given on the figure 27.

9.2 Analytical solutions in simple cases

(294) with $\alpha = 0$ is a convection equation:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0 \tag{296}$$

Material	*diffusion coefficient in water at 25°C, α m²/s*	*Solvability g/100g water*
$CaCl_2$	$1{,}49\cdot 10^{-9}$	42,7
KCl	$2{,}47\cdot 10^{-9}$	25,5
NaCl	$1{,}99\cdot 10^{-9}$	26,3
Cl_2 gas	$1{,}25\cdot 10^{-9}$	0,7
Urine	$1{,}67\cdot 10^{-9}$	

Figure 27: Diffusion coefficients and solvability.

For constant velocity e.g. u = 1 m/s this is an equation of hyperbolic type:

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} = 0 \tag{297}$$

Differentiating with respect to space gives:

$$\frac{\partial^2 C}{\partial t \partial x} + \frac{\partial^2 C}{\partial x^2} = 0 \tag{298}$$

Differentiating with respect to time gives:

$$\frac{\partial^2 C}{\partial t^2} + \frac{\partial^2 C}{\partial x \partial t} = 0 \tag{299}$$

Eliminating the mixed second derivatives results in $\frac{\partial^2 C}{\partial t^2} - \frac{\partial^2 C}{\partial x^2} = 0$, which is the differential equation of a vibrating string. The general solution is $C(x,t) = f(x-t)$, f being a free differentiable function.

Lets suppose the initial condition of a triangle function at the origin for $t = 0 \rightarrow C(x,0) = f(x)$:

$f(x) = 0$, if $x \leq -0,2$

$f(x) = 1 + \frac{x}{0{,}2}$, if $-0,2 \leq 0$

$f(x) = 1 - \frac{x}{0{,}2}$, if $0 \leq 0,2$

Finally, $f(x) = 0$, if $x \geq 0,2$

The solution can be seen on the figure 28.

The 1D diffusion-equation in liquid at rest is:

$$\frac{\partial C}{\partial t} = \alpha \frac{\partial^2 C}{\partial x^2} \tag{300}$$

Defining a new independent variable $\eta = \frac{x}{\sqrt{2\alpha t}}$, C will depend on η: $C = C(\eta)$.

Thus:

$$\frac{\partial C}{\partial t} = \frac{dC}{d\eta}\frac{\partial \eta}{\partial t} = \frac{dC}{d\eta}\left(-\frac{1}{2}\right)\frac{x}{\sqrt{2\alpha t}t} = -\frac{dC}{d\eta}\frac{\eta}{2t} \tag{301}$$

Similarly:

$$\frac{\partial^2 C}{\partial x^2} = \frac{d^2 C}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{d^2 C}{d\eta^2}\frac{1}{2\alpha t} \tag{302}$$

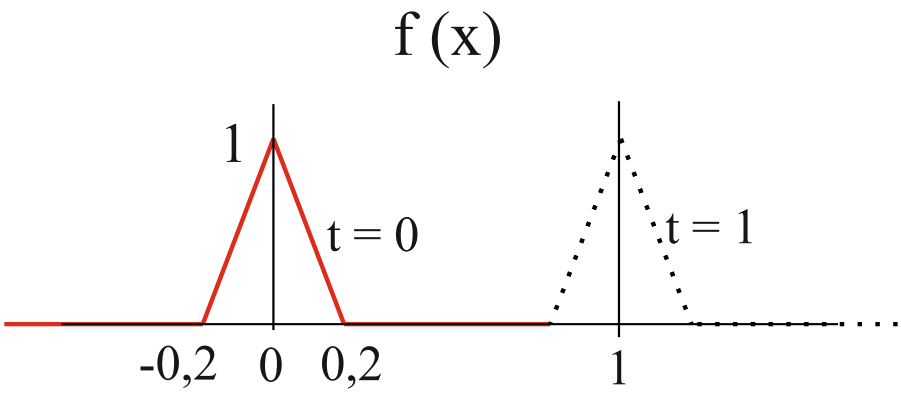

Figure 28: The solution of the DE of vibrating string.

Putting these into (300) gives:

$$-\frac{dC}{d\eta}\eta = \frac{d^2C}{d\eta^2} \tag{303}$$

With the new variable $p = \frac{dC}{d\eta}$ we get $\frac{dp}{d\eta} = -p\eta$. This can be solved easily: $p = Ke^{-\frac{\eta^2}{2}}$ and from this $C = K\int_{-\infty}^{\eta} e^{-\frac{\eta^2}{2}} d\eta + L$. We can notice here the Gaussian distribution function. For a special initial condition $(C = 0 for x > 0)$ the graphs of the consecutive time steps are shown in the figure below computed by $C(x,t) = C_0 2[1 - \Phi(\eta)] = C_0 2\left[1 - \Phi\left(\frac{x}{\sqrt{2\alpha t}}\right)\right]$, $\Phi(\eta)$ being the standard normal distribution function.

For $(t1 < t2 < t3)$ 29 shows the $C_0(x)$ functions.

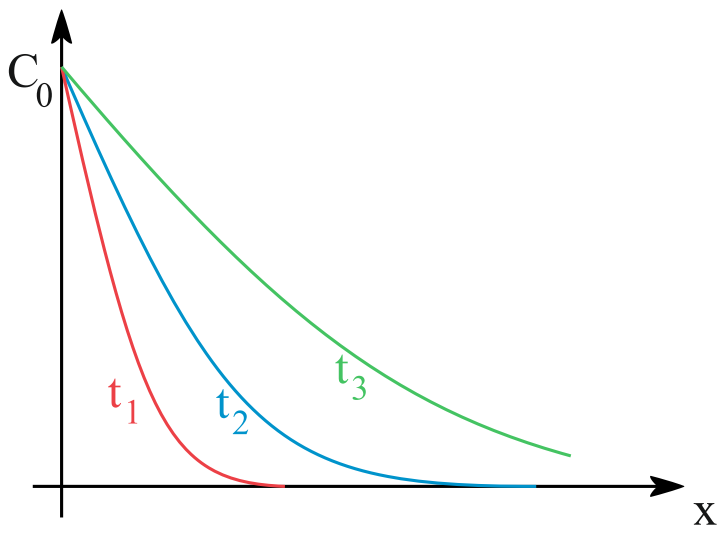

Figure 29: C_0 as a function of x.

$C = 0,5C_0$ if $\eta = \frac{x}{\sqrt{2\alpha t}} = 0,67$. For time $t_1 = 1s$ $x_1 = 0,67\sqrt{2\alpha}$, for $t_2 = 100s$ $x_1 = 6,7\sqrt{2\alpha}$, and for $t_3 = 10000s$ $x_1 = 67\sqrt{2\alpha}$, thus $0,5C_0$ of initial C_0 NaCl-concentration propagates in water at rest during 10000 s (more than 3 hours) to a distance 4,23 mm.

9.3 Numerical discretization

The space is denoted by the subscript i and the time by superscript n. The simplest explicit numerical formula for the time derivative is (forward time = FT):

$$\frac{\partial C}{\partial t} \approx \frac{C_i^{n+1} - C_i^n}{\triangle t} \tag{304}$$

The first and second derivatives with respect to space can be approximated with second order numerical schemes (centred space = CS):

$$\frac{\partial C}{\partial x} \approx \frac{C_{i+1}^n - C_{i-1}^n}{2 \triangle x} \qquad \frac{\partial^2 C}{\partial x^2} \approx \frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{\triangle x^2} \tag{305}$$

The upwind approximation of the first derivative with respect to space is:

$$\frac{\partial C}{\partial x} \approx \frac{C_i^n - C_{i-1}^n}{\triangle x} \tag{306}$$

With these (296) can be approximated with the FTCS scheme:

$$\frac{C_i^{n+1} - C_i^n}{\triangle t} + u\frac{C_{i+1}^n - C_{i-1}^n}{2 \triangle x} = 0 \tag{307}$$

Applying an implicit, backward step in time gives:

$$\frac{C_i^{n+1} - C_i^n}{\triangle t} + u\frac{C_{i+1}^{n+1} - C_{i-1}^{n+1}}{2 \triangle x} = 0 \tag{308}$$

This results in a stable solution.

Applying the upwind scheme we have

$$\frac{C_i^{n+1} - C_i^n}{\triangle t} + u\frac{C_i^n - C_{i-1}^n}{\triangle x} = 0 \tag{309}$$

This difference equation has a conditionally stable solution.

The diffusive term helps to stabilize the solution of the numerical scheme. With FTCS approximation from (293):

$$\frac{C_i^{n+1} - C_i^n}{\triangle t} + u\frac{C_{i+1}^n - C_{i-1}^n}{2 \triangle x} = \alpha\frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{\triangle x^2} \tag{310}$$

After introducing parameters $\sigma = \frac{u\triangle t}{\triangle x}$ and $\beta = \frac{\alpha\triangle t}{\triangle x^2}$ we get:

$$C_i^{n+1} - C_i^n + \frac{\sigma}{2}(C_{i+1}^n - C_{i-1}^n) = \beta(C_{i+1}^n - 2C_i^n + C_{i-1}^n) \tag{311}$$

Explicit schemes are easy to solve starting from the initial concentration distribution. Their drawback is the conditional stability, the time step must be below a limit.
Implicit schemes are stable, but a system of equations must be solved in each time step. As the matrix of the equation-system is sparse (is of diagonal type) this takes not too much time. But there is a danger in applying implicit schemes for centred space approximations. The first derivative is computed from e.g. odd indexed values (empty circles) in even points and of even indexed values (filled circles) in odd points. Thus the sets of even and of odd points are decoupled. We may get a solution like 30.

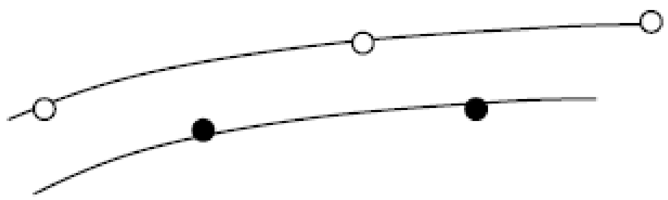

Figure 30: Decoupling of sets of even and odd points.

9.4 Initial condition

A possible initial concentration distribution is a polluted fluid domain in clear liquid. The shape may be put together from three parts, a cosine function for the concentration rise, a constant plateau and another cosine function for the concentration fall.
$C = 0$, if $x \leq -a - r$ or $x \geq a + r$
$C = \frac{C_0}{2}\left(1 + \cos\left(\pi \frac{x+a}{r}\right)\right)$, if $-a - r \leq x \leq -a$
$C = C_0$, if $-a \leq x \leq a$
$C = \frac{C_0}{2}\left(1 + \cos\left(\pi \frac{x-a}{r}\right)\right)$, if $a \leq x \leq a + r$

9.5 Boundary conditions

Boundary conditions of the first kind (Dirichlet) or of the second kind (Neumann) are mainly used. The Dirichlet condition says that the value of the concentration is prescribed on the boundary, e.g, $C = 0$ at the upstream boundary. The Neumann condition prescribes the space derivative of concentration, e.g. $\frac{\partial C}{\partial x} = C'$ at the downstream boundary.

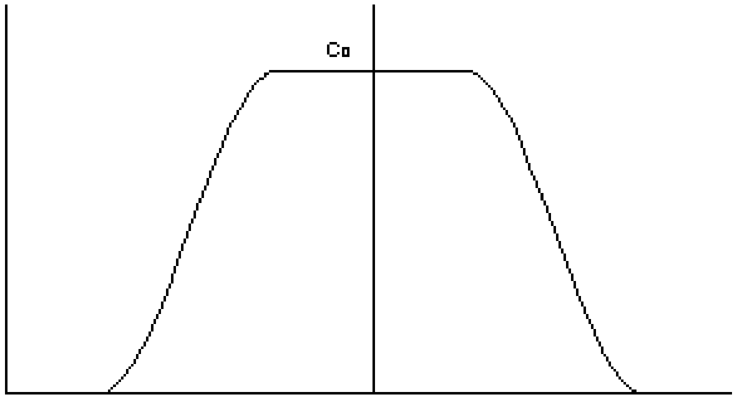

Figure 31: Initial condition.

In inner points centred or upwind schemes can be applied in (293). If the Neumann type condition is prescribed at downstream end then the concentration may be interpolated ($C = ax^2 + bx + c$) through three neighbouring points. For $h = \triangle x$, the point are -h, 0, h with values C_{i-1}, C_i, C_{i+1}. We can check that $C = \frac{C_{i-1}-2C_i+C_{i+1}}{2h^2}x^2 + \frac{C_{i+1}-C_{i-1}}{2h}x + C_i$. From here after differentiation $\frac{\partial C}{\partial x} = \frac{C_{i-1}-2C_i+C_{i+1}}{h^2}x + \frac{C_{i+1}-C_{i-1}}{2h}$.
The boundary condition is the prescribed C value of this derivative.

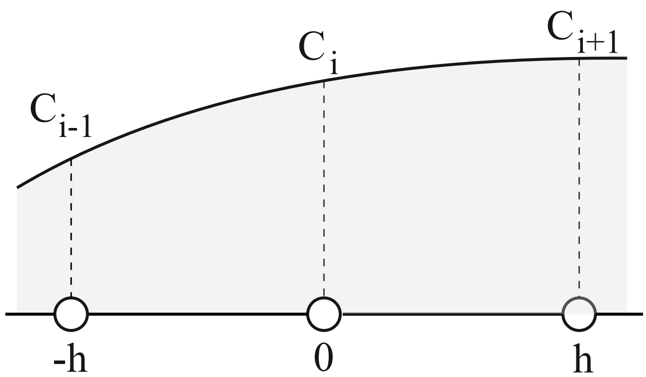

Figure 32: Points and values for boundary conditions.

If $C' = 0$ at $x7 = h$ then $0 = \frac{C_{i-1}-2C_i+C_{i+1}}{h} + \frac{C_{i+1}-C_{i-1}}{2h}$, from here, $C_{i+1} = \frac{4C_i-C_{i-1}}{3}$.
Thus the unknown boundary value of concentration can be expressed with the neighbouring points. For 2D problems (e.g. for flow in a channel of rectangular cross section) both convective and diffusive transport perpendicular to the channel wall is zero. The unit normal vector of the wall is n. There is neither liquid flow through the wall ($q_{k,normal} = wnC = 0C = 0$), nor diffusive transport through the

solid wall ($q_{d,normal} = -\alpha gradCn = -\alpha \frac{\partial C}{\partial n} = 0$).

9.6 Similarity of transport phenomena

The nondimensional form of (293) is:

$$\frac{C_0}{T}\frac{\partial C^*}{\partial t^*} + U\frac{C_0}{L}u^*\frac{\partial C^*}{\partial x^*} = \alpha\frac{C_0}{L^2}\frac{\partial^2 C^*}{\partial x^{*2}} \tag{312}$$

Here we have $x = Lx^*$,$u = Uu^*$,$C = C_0C^*$,$t = Tt^*$. The magnitudes denoted with $*$ are nondimensional. L is the length of the tube, U is the mean liquid velocity, C_0 is the concentration at the plateau of initial distribution, time T is not yet fixed. Dividing by $\alpha\frac{C_0}{L^2}$ we get $\frac{L^2}{T\alpha}\frac{\partial C^*}{\partial t^*} + \frac{LU}{\alpha}u^*\frac{\partial C^*}{\partial x^*} = \frac{\partial^2 C^*}{\partial x^{*2}}$. Here $P = \frac{LU}{\alpha}$ is the Pclet number, $\frac{L^2}{T\alpha}$ has not been definied yet.

There are two possibilities:

- If convection is negligible (U is very small or zero) the P number is small. We can select the time scale to $T_1 = \frac{L^2}{\alpha}$ thus the process depends on one single parameter:

$$\frac{\partial C^*}{\partial t^*} + P \quad u^*\frac{\partial C^*}{\partial x^*} = \frac{\partial^2 C^*}{\partial x^{*2}} \tag{313}$$

- If diffusion is negligible compared to convection $T_2 = \frac{L^2}{\alpha}$, again the Pclet number is characterizing the process and it is very large thus:

$$P\frac{\partial C^*}{\partial t^*} + P \quad u^*\frac{\partial C^*}{\partial x^*} = \frac{\partial^2 C^*}{\partial x^{*2}} \tag{314}$$

10 Conservation laws

10.1 Introduction

In most engineering applications, the behaviour of physical systems are governed by conservation laws. Suppose that we have a physical domain Ω with boundary $\partial\Omega$. Then, the *integral form* of a conserved scalar quantity U is

$$\frac{d}{dt}\int_{(\Omega)} U d\Omega + \int_{\partial\Omega} \underline{F}(U)\cdot\underline{n}\, ds = \int_{(\Omega)} S(U,t)d\Omega \tag{315}$$

where $\mathbf{F}(U)$ is the flux vector, $\mathbf{n}$ is the unit normal vector pointing outward from the boundary and S is the source term. For example, let $U = \rho$ be the density and consider the domain to be an 1D pipe with length $0 \leq x \leq L$ and varying cross section $A(x)$. Then, $\int_0^L A(x)\rho(x)dx$ is the total mass in the system and

$$\frac{d}{dt}\int_0^L A(x)\rho(x)dx = \frac{d}{dt}m(t) \tag{316}$$

is the rate of change of the mass inside the domain. To evaulate the second term, we have to define the 'flux function', which, in the case of mass conservation is $F(U) = \rho v$, with v being the velocity. Let us assume that the only surfaces on which flow-through is allowed is at the two ends of the pipe, then

$$\int_{\partial\Omega} \underline{F}(U)\cdot\underline{n}\, ds = \rho(L)v(L)A(L) - \rho(0)v(0)A(0) = \dot{m}_{out} - \dot{m}_{in}. \tag{317}$$

Note that $\dot{m}$ denotes mass flow rate (kg/s) and $m(t)$ denotes the mass within the domain (that might change in time).

In general, for an 1D conservation law, the *conservative form* reads

$$\frac{d}{dt}\int_a^b U dx + F(U(b,t)) - F(U(a,t)) = 0. \tag{318}$$

The divergence theorem states that

$$\int_{\partial\Omega} \underline{F}\cdot\underline{n}\, ds = \int_{(\Omega)} \nabla\cdot\underline{F}\, d\Omega. \tag{319}$$

The LHS represents the fluxes entering or leaving the domain, while the RHS (divergence) measures the 'source strength' (sink strength) inside the domain. For example, consider the velocity field

$$v_x(x) = x. \tag{320}$$

This is symmetrical around $x = 0$ and it is clear that $v_x > 0$ on the right half-plane ($x < 0$) and negative on the left plane. Consider the domain $-L/2 \leq x \leq L/2$. The LHS is

$$\int_{\partial\Omega} \underline{F} \cdot \underline{n}\, ds = v|_{x=-L/2} \times \underbrace{(-1)}_{\underline{n}_L} + v|_{x=L/2} \times \underbrace{1}_{\underline{n}_R} = (-L/2) \times (-1) + L/2 = L. \tag{321}$$

The divergence is

$$\int_{(\Omega)} \nabla \cdot \underline{F}\, d\Omega = \int_{-L/2}^{L/2} \frac{\partial v_x}{\partial x} dx = \int_{-L/2}^{L/2} 1 dx = L. \tag{322}$$

The Nabla operator is $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, hence $\text{div}(F) = \nabla \cdot F$.

Turning back to the conservation law, upon making use of the divergence theorem, the surface integral can be rewritten as a volume integral:

$$\frac{d}{dt} \int_{(\Omega)} U d\Omega + \int_{(\Omega)} \nabla \cdot \underline{F}\, d\Omega = \int_{(\Omega)} S(U,t) d\Omega \quad \text{(conservative form)} \tag{323}$$

or

$$\frac{dU}{dt} + \nabla \cdot \underline{F} = S(U,t) \quad \text{(differential form)}. \tag{324}$$

Note that, upon integrating the conservative form between (arbitrary) $t_- < t_+$, we have

$$\int_a^b U(x,t_+) = \int_0^b U(x,t_-) - \left(\int_{t_-}^{t_+} F(U(b,t))dt - \int_{t_-}^{t_+} F(U(a,t))dt \right), \tag{325}$$

which is derivative-free.

10.2 Linear transport equation and concept of characteristics

Consider the following simple transport equation

$$u_t + cu_x = 0, \quad u(x,0) = g(x). \tag{326}$$

Interestingly, $u(x,t) = g(x - ct)$ solves this problem, as

$$u_t = -cg'(x - ct) \quad \text{and} \quad u_x = g'(x - ct). \tag{327}$$

Moreover,

$$u(x,t) = g(x-ct) = g((x-ct)-0c) = u(x-ct,0), \tag{328}$$

which means that the solution at location x and time t is the same as the one at location $x-ct$ at $t=0$. Hence, this solution is the uniform translation of $g(x)$ along the x axis with velocity c.

Now consider the classic second-order wave equation

$$u_{tt} - c^2 u_{xx} = 0. \tag{329}$$

Let us introduce the new variable $v = u_t + cu_x$, then we have

$$v_t = u_{tt} + cu_{xt} = \underbrace{c^2 u_{xx}}_{\text{from (329)}} + cu_{xt} = c\,(cu_x + u_t)_x = cv_x. \tag{330}$$

Hence, instead of the (329) wave equation, we have to solve

$$v_t - cv_x = 0 \quad \text{and} \tag{331}$$

$$u_t + cu_x = v. \tag{332}$$

The solution of the first problem – as we have already seen – is $h(x+ct)$. For the second, inhomogenuous problem, the solution of the homogenous problem is

$$u_{\text{hom}} = g(x-ct). \tag{333}$$

The guess solution of the inhomogenuous problem ($u_t + cu_x = h(x+ct)$) will be $u_p = f(x+ct)$, which gives

$$\underbrace{f'c}_{u_t} + c\underbrace{f'}_{u_x} = h(x+ct) \quad \to \quad f(s) = \frac{1}{2c}\int h(s)ds. \tag{334}$$

We finally see that the general solution of the second-order wave equation is the sum of two waves

$$u(x,t) = f(x+ct) + g(x-ct). \tag{335}$$

10.3 1D unsteady gas dynamics

The governing equations are

$$(\rho A)_t + (\rho A v)_x = 0 \tag{336}$$

$$(\rho A v)_t + \left(\rho A v^2 + pA\right)_x = p\frac{dA}{dx} + F_s \tag{337}$$

$$(\rho A E)_t + (\rho A v H)_x = \dot{Q} \tag{338}$$

where $E = e + v^2/s$ is the total energy ($e = c_v T$) and $H = E + p/\rho = c_p T + v^2/2$ is the total enthalpy. We also have

$$\left.\frac{dp}{d\rho}\right|_s = c^2, \tag{339}$$

$$T ds = dh - \frac{dp}{\rho} = de + pd\frac{1}{\rho} \rightarrow \left.\frac{dh}{dp}\right|_s = \frac{1}{\rho}, \left.\frac{de}{dp}\right|_s = \frac{p}{\rho^2 c^2}. \tag{340}$$

For $A = A(x)$ (that is, the channel cross section does not vary in time), the differential form of the above equations are as follows.

$$\begin{aligned} (\rho A)_t + (\rho A v)_x &= 0 \\ \rho \cancelto{0}{A_t} + \rho_t A + \rho A v_x + \rho A_x v + \rho_x A v &= 0 \\ \underbrace{\rho_t}_{①} + \underbrace{\rho v_x}_{②} + \underbrace{v \rho_x}_{③} &= \underbrace{-\frac{\rho v}{A} A_x}_{④} \end{aligned} \tag{341}$$

$$\begin{aligned} (\rho A v)_t + \left(\rho A v^2 + pA\right)_x &= p\frac{dA}{dx} + F_s \\ \rho A v_t + \underbrace{\rho_t A v}_{Av①} + \underbrace{\rho_x A v^2}_{Av③} + \underbrace{\rho A_x v^2}_{Av④} + \underbrace{\rho A 2 v v_x}_{2Av②} + p_x A + \cancel{pA_x} &= \cancel{pA_x} + F_s \\ \underbrace{v_t}_{⑤} + \underbrace{v v_x}_{⑥} + \underbrace{\frac{1}{\rho} p_x}_{⑦} &= \frac{F_s}{\rho A} \end{aligned} \tag{342}$$

$$
\begin{aligned}
(\rho A E)_t + (\rho A v H)_x &= \dot{Q} \\
\rho_t A E + \rho A E_t + & \\
\rho_x A v H + \rho A_x v H + \rho A v_x H + \rho A v H_x &= \dot{Q} \\
\frac{1}{c^2} p_t A E + \rho A \left(\frac{p}{\rho^2 c^2} p_t + v v_t \right) + & \\
\frac{1}{c^2} p_x A v H + \rho A_x v H + \rho A v_x H + \rho A v \left(\frac{p}{\rho^2 c^2} p_t + v v_t + \frac{1}{\rho} p_t - p \frac{\rho_x}{\rho^2} \right) &= \dot{Q} \\
\text{... some steps skipped here ...} & \qquad (343) \\
p_t + v p_x + \rho c^2 v_x &= \frac{\dot{Q}}{H} - \frac{\rho v c^2}{A} A_x \qquad (344)
\end{aligned}
$$

The quasilinear form is

$$
U_t + \mathcal{A}(U) U_x = S \quad \text{with} \quad U = \begin{pmatrix} \rho \\ v \\ p \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} v & \rho & 0 \\ 0 & v & 1/\rho \\ 0 & \rho c^2 & v \end{pmatrix}. \tag{345}
$$

We also have

$$
\mathcal{A}^{-1} = \begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & c\rho/2 & 1/2 \\ 0 & -c\rho/2 & 1/2 \end{pmatrix}. \tag{346}
$$

and the eigenvalues and eigenvectors are

$$
\lambda_1 = v \qquad v_1 = (1, 0, 0)^T \tag{347}
$$

$$
\lambda_2 = v + c \quad v_2 = \left(\frac{1}{c^2}, \frac{1}{c\rho}, 1 \right)^T \tag{348}
$$

$$
\lambda_3 = v - c \quad v_3 = \left(\frac{1}{c^2}, -\frac{1}{c\rho}, 1 \right)^T \tag{349}
$$

The new variables diagonalizing the equatios are

$$
V = \mathcal{A}^{-1} U = \begin{pmatrix} \rho - \frac{p}{c^2} \\ \frac{1}{2}(p + \rho c v) \\ \frac{1}{2}(p - \rho c v) \end{pmatrix} \tag{350}
$$

and we have $\rho - \frac{p}{c^2} = \rho - \frac{p}{\kappa R T} = \rho - \frac{\rho}{\kappa} = \frac{\kappa - 1}{\kappa} \rho$.